Alternative explanation for dual expectation

Consider a simple case where N=2 (result can be generated to arbitrary N) and an information structure G defined by its cutoffs $\{0, s_1, 1\}$. This information structure induces two posterior means, x_1 and x_2. Now we construct a minorant function $\underline{u}$ function as below, [1]

$$\underline{u}(x) = \begin{cases} u(x_1) + u'(x_1)(x - x_1) & \text{if } x \in [0, s_1), \\ u(x_2) + u'(x_2)(x - x_2) & \text{if } x \in [s, 1]. \end{cases} \tag{1}$$

Therefore, $\underline{u}(x)$ is convex and is piece-wise affine. Moreover, it is tangent to $u(x)$ at each posterior mean. We want to argue that $\underline{u}$ is also continuous at s_1 under the optimal solution, then the dual expectation such that s_1 is the conditional expectation on $[x_1, x_2)$ under the distribution u' follows from Corollary 1.

We prove by contradiction. Suppose the information structure G, characterized by the interval cutoffs $\{0, s_1, 1\}$, is optimal and the minorant function $\underline{u}$ constructed accordingly is not continuous at s_1, as in the left panel of Figure 1. Then consider another function $\underline{\hat{u}}$ such that,

$$\underline{\hat{u}}(x) = \max\{u(x_1) + u'(x_1)(x - x_1), u(x_2) + u'(x_2)(x - x_2)\},$$

where $u(x_1) + u'(x_1)(x - x_1)$ meets $u(x_2) + u'(x_2)(x - x_2)$ at $\hat{s}_1 \in (x_1, x_2)$. See the right panel of Figure 1. Note that $\hat{s}_1$ exists. We also define the corresponding information structure $\hat{G}$, characterized by the interval partition with cutoffs $\{0, \hat{s}_1, 1\}$.

Then the following inequality holds.

$$\begin{aligned} \int u \mathrm{d}G = \int \underline{u} \mathrm{d}G &= \int_0^{s_1} \underline{u} \mathrm{d}F + \int_{s_1}^1 \underline{u} \mathrm{d}F \\ &\leq \int_0^{\hat{s}_1} \underline{\hat{u}} \mathrm{d}F + \int_{\hat{s}_1}^1 \underline{\hat{u}} \mathrm{d}F = \int \underline{\hat{u}} \mathrm{d}\hat{G} \leq \int u \mathrm{d}\hat{G}. \end{aligned} \tag{2}$$

The first equality is because the information structure G puts mass points at x_1 and x_2. The second equality comes from that $\underline{u}$ is piece-wise linear in state. The first inequality

[1] In a single-agent decision problem, each line segment of $\underline{u}$ represents the DM's payoff in state θ when he takes the optimal action $y^*(x_i)$.

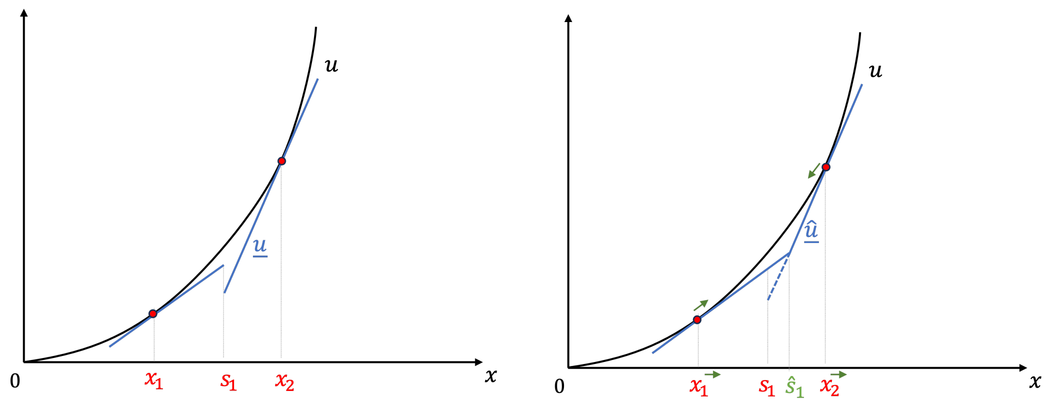

Figure 1: Contradiction

in the second row comes from $\underline{\hat{u}}$ is everywhere above $\underline{u}$. The equality in the second row again comes from the piece-wise linearity of $\underline{\hat{u}}$. The last inequality is because u is everywhere above $\underline{\hat{u}}$. That is, there exists another information structure $\hat{G}$ better than G if $\underline{u}$ is not continuous at s_1.[2]

[2]We thank Gregorio Curello for his inspiration of this proof.