

Coarse Information Design*

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Abstract: We study an information design problem with continuous state and discrete signal space. Under convex value functions, the optimal information structure is interval-partitional and exhibits a dual expectations property: each induced signal is the conditional mean (taken under the prior density) of each interval; each interval cutoff is the conditional mean (taken under the value function curvature) of the interval formed by neighboring signals. This property enables examination into which part of the state space is more finely partitioned and facilitates comparative statics analysis. The analysis can be extended to general value functions and adapted to study coarse mechanism design.

Keywords: dual expectations, scrutiny, S-shaped value function, coarse non-linear pricing, energy efficiency ratings

JEL Classification: D81, D82, D83

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1. Introduction

Information is critical in decision making. In single-agent decision problems, if any experiment is feasible and costless, it is optimal to choose one that fully reveals the state. However, perfectly-revealing experiments are oftentimes infeasible. In many situations information takes a discrete form. For example, teachers evaluate students by letter grades or on a pass/fail basis; Michelin Guide ranks the recommended restaurants up to three stars for excellence; online platforms often convey product reviews through coarse rating systems; and regulatory policies disclose efficiency and safety level of different products via discrete categorizations. In this paper, we take the discrete nature of information structure as given and study how to coarsen information optimally subject to such constraint.

Coarse information emerges naturally due to limited cognitive and memory capacities, imperfect communication channels, technological constraints on the measurement instruments, or simply for the sake of convenience. In pain medicine, for example, the intensity of pain naturally falls into a continuum, but it would be almost impossible for patients to fully describe their subjective feeling or convert it into a real number on a continuous scale. Instead, doctors rely on instruments such as the Numerical Rating Scale, from 0 to 10, based on patients' self-reported pain level to reach a diagnosis. In an organizational context, individuals at different levels of the hierarchy may not have the time or the expertise to digest complicated information from each other if the communication involves exhaustive details. In practice, they typically adopt language protocols that are less precise but more comprehensible to facilitate decision-making. Furthermore, because different individuals in the organization are making different decisions, the optimal way to coarsen information is tailored to different needs. An executive summary of a research report written for the head of engineering, for example, should be quite different from a summary of the same report written for the CEO.

Formally we study a class of information design problems where the uncertain state θ is continuously distributed on $[0, 1]$ and is payoff relevant only through its expectation. The (information) designer—or the sender—commits to a Blackwell experiment (π, Σ) , where Σ is the signal space and $\pi(\sigma|\theta)$ is the probability of obtaining signal realization $\sigma \in \Sigma$ conditional on state θ . The receiver updates her belief based on the realized signal and then chooses an action optimally. The constraint we introduce in this paper is that Σ is restricted to be a finite set with cardinality of at most N . Given

that the state is continuous and the signals are discrete, an experiment necessarily involves pooling across different states, and some degree of information loss is inevitable. How to allocate limited “signal resources” then becomes a nontrivial question. From a design perspective, an economically relevant question emerges: Which parts of the state space should receive *closer scrutiny* in the experiment relative to other parts? In this introduction, unless otherwise specified, we illustrate the issues involved and the results under the assumption that the designer’s value function (as a function of the induced posterior mean) is convex. Nevertheless, all results remain true for arbitrary S-shaped value functions.

Consider an *interval-partitional* information structure (the optimality of which will be established in this paper), where an experiment divides the state space into a finite number of subintervals and reveals which subinterval the state belongs to. We say that a subinterval of the state space receives “closer scrutiny” than others if the width of this subinterval is smaller than that of others. In Figure 1, we represent an experiment by the set of cutoffs that defines the partition. Under experiment *A*, the state space in the neighborhood of 0.7 receives the closest scrutiny. In contrast, experiment *B* explores the state space near 0.3 with the closest scrutiny. We refer to the subinterval that receives the closest scrutiny as the “center of scrutiny.”

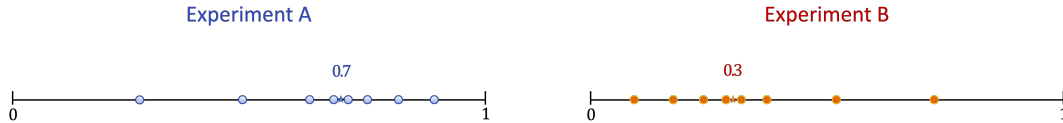


Figure 1: Experiment *A* gives close scrutiny to the state space near 0.7; experiment *B* gives close scrutiny to the state space near 0.3.

The location of the scrutiny center is jointly determined by both the prior distribution $F(\cdot)$ and the designer’s interim value function $u(\cdot)$. Naturally, the designer should give closer scrutiny to states that are more likely to happen, because having the receiver make correct decisions under those states carries a greater weight in the designer’s ex ante expected utility.¹ At the same time, the designer should give closer scrutiny to those states where making informative decisions is more valuable, i.e., where the value

¹Notice that when $u(\cdot)$ is convex, the designer and the receiver both prefer more information to less information.

function has greater curvature, measured by the magnitude of the second derivative u'' . We show that the optimal experiment generally reflects these two concerns.

Figure 1 highlights another feature of the optimal information structure in a *log-concave* environment (i.e., one in which both the prior density f and the curvature u'' are logconcave); namely, the widths of the subintervals are *single-dipped*. Specifically, there is a center of scrutiny—a subinterval that receives the closest scrutiny. As we move away from this center, the level of scrutiny gradually diminishes for states located farther away.

In Section 3, we characterize the designer’s optimal information structure and establish a dual relation between the prior density f and the curvature u'' . Under the optimal experiment, the state space is divided into N subintervals denoted as $[s_{k-1}, s_k]$, where $s_0 = 0$ and $s_N = 1$. The k -th interval only induces one signal x_k , which is equal to the conditional mean of the states between s_{k-1} and s_k , taken with respect to the conditional density $f(\cdot)/(F(s_k) - F(s_{k-1}))$. Remarkably, the optimal experiment exhibits a dual property: the cutoff point s_k is equal to the “expectation” of the interval $[x_k, x_{k+1}]$, taken with respect to the “conditional density” $u''(\cdot)/(u'(x_{k+1}) - u'(x_k))$. Moreover, all the results we derive concerning the set of interval cutoffs can be extended to the set of signals (represented by the posterior means). For example, in a logconcave environment, the widths between two adjacent posterior means also display the single-dipped property.

The dual expectations property enables us to characterize the optimal experiment as the fixed point of a monotone system of equations, which then facilitates our comparative statics analysis. We find that a likelihood-ratio increase in the prior density f or the curvature function u'' results in a rightward shift of all the interval cutoff points (s_1, \dots, s_{N-1}) as well as the posterior means (x_1, \dots, x_N) . To further analyze the effects of variability changes in the prior density and value function, we adopt the notion of *uniform conditional variability order* (Whitt, 1985). We find that as f or u'' becomes less variable according to this order, the conditional means of the optimal experiment becomes more compressed, in the sense that there exists an i^* such that x_i shifts to the right if $i < i^*$ and x_i shifts to the left if $i > i^*$.

The duality property also allows us to adapt the analysis to study mechanism design where the agent’s payoff is linear in his private information and the contract is constrained to be finite. In Section 3.3, we present an example to illustrate how to

transform a standard nonlinear pricing problem with finite menu into a coarse information design problem such that most of our results are transferable to study the design and properties of the optimal discrete menu.

Section 4 extends the analysis to address S-shaped value functions.² We show that the optimal experiment is still interval-partitional, and is always less Blackwell-informative than the optimal experiment without the discreteness constraint. Moreover, the comparative statics results for the case of convex utility functions carry over readily to S-shaped utility functions. Section 5 further summarizes the toolbox to study the optimal information structure under general value functions, where bi-pooling policies are potentially relevant. We show that the dual expectations property remains valid, incorporating the potential usage of bi-pooling policy through a similar bi-tangency condition as in Dworczak and Martini (2019) and Arieli et al. (2023).

We provide further discussions of the model in Section 6. Section 6.1 studies the design of energy efficiency ratings system as an application of our model, and provides a rationale for a commonly observed pattern in such rating standards. Section 6.2 uses numerical examples beyond logconcave environments to demonstrate the robustness of our conclusions regarding the center of scrutiny. In Section 6.3 we establish connections between our model and cheap talk, considering both exogenous and endogenous information structures.

Related literature. Our paper contributes to the existing literature on information design with a continuous state space. We adopt the approach introduced by Gentzkow and Kamenica (2016), which represents an information structure by the integral of its cumulative distribution function. Kolotilin (2018) and Dworczak and Martini (2019) use alternative approaches to studying this problem. Kleiner et al. (2021) and Arieli et al. (2023) show that the optimal unconstrained information structure exhibits the bi-pooling property. Recently, Curello and Sinander (2023) study the comparative statics of general linear persuasion problems. They identify the conditions under which a sender with “more convex” value function will design a more informative signal structure. The bulk of the Bayesian persuasion literature starting from Kamenica and Gentzkow (2011) and Rayo and Segal (2010) is concerned about the strategic use of pooling: how the sender designs an experiment to “concavify” her value function by strategically pooling information in a way to influence the receiver’s action. When the

²S-shaped value function is well-studied in Bayesian persuasion literature, e.g., Kolotilin et al. (2022).

signal space is constrained to be discrete, pooling becomes necessary even when the value function is convex. Our paper studies a constrained information design problem and examines how to effectively pool information in such environment.

The constraint imposed by coarse information on the operation of markets is discussed by Wilson (1989), McAfee (2002) and Hoppe et al. (2010). Dow (1991) considers a sequential search problem for a decision maker whose memory is represented by a partition of the set of possible past prices. Our paper generalizes the insight into a broader class of design problems. Harbaugh and Rasmusen (2018) study how coarse grading can increase information by raising the incentive to get costly certification; while Ostrovsky and Schwarz (2010) analyze coarse grading from an information design perspective. Lipman (2009) points out the efficiency loss from the vagueness of language, and Crémer et al. (2007) study the optimal coarse language code under a discrete environment where the loss function only depends on the number of states that are pooled together in one message. They discuss the implications of coarse language for the theory of the firm and organizational hierarchy. Applying the toolkit of Voronoi diagram, Jäger et al. (2011) study the behavior and stability of the optimal language by a distance-based criterion.

Recently, the literature on information design has begun to explore the limitation of communication and information channels. Onuchic and Ray (2022) and Mensch (2021) study information design subject to monotone information structures. Aybas and Turkel (2022) examine an information design problem under a bounded signal space. They characterize the highest achievable sender payoff and analyze how it changes with the cardinality of the signal space. Their paper explores finite state and finite action space,³ and allows the sender’s payoff to depend arbitrarily on beliefs. In contrast, we focus specifically on linear persuasion problems and provide a general characterization for the optimal information structure under convex and S-shaped value functions.⁴ Furthermore, our paper is closely related to Hopenhayn and Saeedi (2022), which studies the optimal coarse ratings system that partitions a continuum of sellers with different qualities into a finite number of quality groups in a competitive environment. They compare the rating schemes under different shapes of supply functions.

³Le Treust and Tomala (2019) and Doval and Skreta (2023) develop powerful machinery to study persuasion problems with general constraints under finite states.

⁴Ivanov (2021) also addresses linear persuasion problems, but he allows the sender’s payoff to depend on the information structure non-strategically, i.e., a non-strategic receiver acts upon both the realized signal and the information structure.

Tian (2022) uses “cell functions” (a mapping from subintervals to payoffs) as primitives to study the optimal interval partition problem. He focuses on submodular cell functions and shows that the optimal interval cutoffs shift up when the prior belief shifts up. In contrast, we use interim value functions as primitives and characterize the optimal discrete information structure. We uncover a dual relation between value function curvature and prior density, and exploit this property to show how the optimal information structure changes as the value function varies.

Similar discrete constraints are also studied in many mechanism design problems with transfers. Bergemann et al. (2012, 2021) and Wong (2014) introduce the discrete constraint in a standard nonlinear pricing problem and discuss the payoff bound of the finite menu and asymptotic convergence rate. In Section 3.3, we establish a connection between these two problems. Recently, Bergemann et al. (2022) study the role of information design in a classic nonlinear pricing problem, demonstrating that the unconstrained optimal information structure is endogenously coarse.

This paper is related to the classic cheap talk model of Crawford and Sobel (1982). Comparative statics in cheap talk models regarding the prior distribution have been explored in Szalay (2012), Chen and Gordon (2015), and Deimen and Szalay (2023). Recently, the literature on cheap talk begins to explore the sender’s optimal design of information. Ivanov (2010) characterizes the optimal information structure in a uniform-quadratic cheap talk game, with a focus on interval-partitional structures. Deimen and Szalay (2019) consider a two-dimensional cheap talk game where the sender has access to a signal structure with elliptical distribution. Lyu and Suen (2023) study information design in cheap talk and characterize the optimal information structure under arbitrary preferences and binary states. Kreutzkamp (2023) and Lou (2023) explore the game under continuum states and posterior-mean dependent payoffs, and prove the optimality of bi-pooling structures. When information is costless, both papers independently characterize the solution under uniform prior and quadratic value functions. In cheap talk games, information coarsening emerges endogenously from incentive compatibility. In our model, information coarsening is imposed as an exogenous constraint. At the end of our paper, we discuss the connection between our research and the classic cheap talk game.

The coarse information design problem addressed in this paper resembles quantization in information theory (Gray and Neuhoff, 1988; Mease and Nair, 2006). These two

classes of problems are not nested. The main difference is that quantization algorithms typically maximize the expected “similarity” within a group based on a distance metric, e.g., minimizing the expected distance between the realized values and the centroid within a cluster, which in general, cannot be transformed into a well-defined interim value function with a domain of posterior means. One exception is when the loss function is quadratic,⁵ in which case the quantization problem becomes equivalent to our problem with uniform value function curvature. In applied econometrics, heuristic procedures such as the equal-division rule (partitioning the state space into equal-width bins) or the equal-probability rule (assigning equal probabilities to observations falling into each bin) are commonly used to categorize continuous variables. In Section 3.1, we discuss how our optimal experiment compares to these heuristic rules. On a more abstract level, our research is also related Robson (2001) and Netzer (2009). These studies explore the impact of coarse perception on the evolution of preferences, while our work focuses on optimal decision-theoretic responses to coarse information.

2. Preliminaries

Consider an information design model where the payoff-relevant state θ is drawn from a prior distribution F on $[0, 1]$. We assume that F admits a continuous density function f with full support. A designer commits to an information structure: a signal space Σ and a mapping $\pi : [0, 1] \rightarrow \Delta(\Sigma)$ from the state to a distribution over signals, which induces a random posterior. We focus on models where the designer’s interim payoff depends on the induced posterior belief only through the posterior mean, and use $u : [0, 1] \rightarrow \mathbb{R}$ to denote this interim value function.⁶ Throughout the paper, we maintain the assumption that u is twice continuously differentiable and regular as in Dworczak and Martini (2019). Since the only relevant information is the posterior mean, it is without loss of generality to assume that the realized signal is the posterior mean itself. From now on, we will primarily work with the induced distribution G of random posterior means.

We follow Gentzkow and Kamenica (2016) by representing an information structure as the integral of the induced distribution function of posterior mean. Let F_0 be the

⁵The least-squares criterion, or mean-squared-error model is the workhorse in the quantization literature. For a classical algorithm, see Lloyd (1982).

⁶As Dworczak and Martini (2019) point out, the posterior-mean dependent setup is satisfied in a sender-receiver environment when the receiver’s optimal action only depends on the expected state and the sender’s preference over action is linear on the state.

degenerate distribution that puts probability mass one on the prior mean of F . For any distribution G on $[0, 1]$, define $I_G(x) := \int_0^x G(t) dt$ and call it the *integral distribution* of G . By Strassen’s (1965) theorem, G can be induced by some signal structure if and only if $I_{F_0} \leq I_G \leq I_F$; namely, G must be a mean-preserving contraction of F .

The main ingredient of our model is the introduction of the discreteness constraints on the signal space Σ . In particular, we require Σ to contain no more than N elements. Consequently, the distribution G of posterior means can only have a finite support.

We can write the designer’s optimization program as:

$$\max_{G \in \Delta([0,1])} \int_0^1 u(x) dG(x) \quad (1)$$

$$\text{s.t. } I_{F_0} \leq I_G \leq I_F, \quad (\text{MPC})$$

$$|\text{supp}(G)| \leq N. \quad (\text{D})$$

As we argue in the introduction, coarse information is ubiquitous in many aspects of economic activities. The discreteness constraint (D) introduces a natural question about how to optimally allocate limited “signal resources” in information design.

A distribution with finite support has an integral distribution which is increasing, convex and piecewise linear, with kinks at every element in the support of the distribution. Let ICPL denote the set of such integral distribution functions defined on $[0, 1]$ that satisfy the mean-preserving contraction constraint (MPC). For every $I \in \text{ICPL}$, define the set of kink points of I by $\mathcal{K}_I := \{x \in (0, 1) : I'(x^-) \neq I'(x^+)\}$.

To better interpret the integral distribution I_G with $|\mathcal{K}_{I_G}| \leq N$, we consider the following class of information structures. Partition the state space into $K \leq N$ subintervals with a sequence of cutoff points, $0 = s_0 < s_1 < \dots < s_K = 1$. Within each subinterval $[s_{k-1}, s_k)$ for $k \in \{1, \dots, K\}$, there are $J_k \geq 1$ signals induced. Denote the induced posterior means by $\{x_k^j\}_{j=1}^{J_k}$, such that $\sum_{k=1}^K J_k \leq N$. Following Kleiner et al. (2021) and Arieli et al. (2023), we call this class of information structures *J-pooling* policies if $\max_k \{J_k\}_{k=1}^K = J$. Since K can be equal to 1, any discrete information structure satisfying (D) is a *J-pooling* policy for some value of J . In the special case of $J = 2$, we call it a *bi-pooling* structure⁷; and if $J = 1$, we call it an *interval-partitional* structure. The

⁷Arieli et al. (2023) show that any bi-pooling policy over an interval can alternatively be implemented by some pure nested-interval policy and by some potentially mixed FOSD-ranked monotone policy.

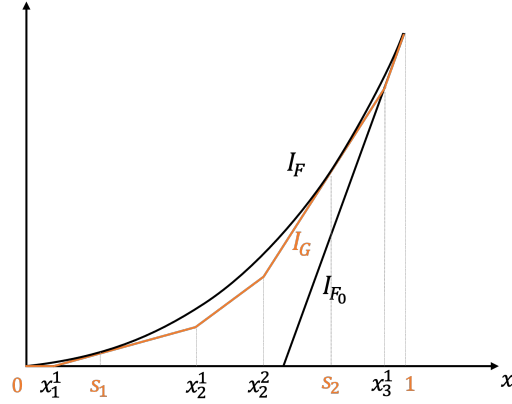


Figure 2: The tangency points are the interval cutoffs and the kink points are the induced posterior means under J -pooling policy.

following lemma identifies the corresponding J -pooling information structure for every ICPL function.

Lemma 1. *For any $I_G \in \text{ICPL}$, the corresponding distribution G can be induced by a J -pooling policy characterized as follows:*

- (i) $\{s_k\}_{k=0}^K$ are the points where I_G is tangent to I_F ;
- (ii) for each $k = 1, \dots, K$, $\{x_k^j\}_{j=1}^{J_k}$ are the kink points of \mathcal{K}_{I_G} that lie in $[s_{k-1}, s_k]$.

In Figure 2, the integral distribution function I_G is tangent to I_F at $\{s_0, s_1, s_2, s_3\}$, where $s_0 = 0$ and $s_3 = 1$. The J -pooling policy derived from I_G sends only one signal (which induces posterior mean x_1^1) if $\theta \in [s_0, s_1]$, and one signal (which induces posterior mean x_3^1) if $\theta \in [s_2, s_3]$. If $\theta \in [s_1, s_2]$, the information structure generates two distinct signals, inducing posterior mean x_2^1 or x_2^2 . For any J -pooling policy with $J > 1$, some segments of the corresponding I_G curve is strictly below I_F . For an interval-partitional structure, on the other hand, every segment of I_G has a tangency point with I_F .

An important implication of Lemma 1 is that if I_G has only one kink point between two adjacent tangency points s_{k-1} and s_k , then the kink point must be the posterior mean conditional on the subinterval $[s_{k-1}, s_k]$. For example, this property holds trivially for I_{F_0} , where the two adjacent tangency points are 0 and 1, and the only kink point is the prior mean $\mathbb{E}_F[\theta]$. The following corollary suggests that this simple property holds generally for any subinterval.

Corollary 1. For any $I_G \in \text{ICPL}$, if $J_k = 1$ for some k , then $x_k^1 = \mathbb{E}_F[\theta \mid \theta \in [s_k, s_{k+1})]$.

We may transform objective function (1) of the optimization problem through integration-by-parts into an explicit integral of an ICPL function:

$$\int_0^1 u(x) dG(x) = u(1) - u'(1)I_F(1) + \int_0^1 u''(x)I_G(x) dx$$

The first two terms are constants. Hence, the original problem can be rewritten as:

$$\begin{aligned} \max_{I_G \in \text{ICPL}} \quad & \int_0^1 u''(x)I_G(x) dx & (2) \\ \text{s.t.} \quad & |\mathcal{K}_{I_G}| \leq N. & (D) \end{aligned}$$

The objective function (2) represents the “signed weighted” area under the integral distribution function, where the signed weights are given by the curvature of the value function, u'' . Intuitively, the level of an integral distribution I measures how informative the corresponding information structure is locally. A higher I is closer to the full information case I_F . Hence, it yields a higher payoff if providing local information is very rewarding, namely, associated with higher positive u'' . Similarly, a lower I is closer to the null information case I_{F_0} , and is more preferable to the designer if u'' is negative. Obviously, if u is convex within some region, a locally upward shift of I_G increases the designer’s expected payoff, provided that such shift does not exceed I_F . Moreover, a clockwise rotation of certain segment of I_G at the point when u switches from convex to concave is beneficial, provided that I_G remains in the ICPL class.

The next result shows that we only need to focus on a much smaller subset of ICPL functions. When searching for an optimal solution, it suffices to consider the bi-pooling information structure. The proof is in the spirit of Kleiner et al. (2021) and Arieli et al. (2023). Some minor modification is required, because although the objective function is still a linear functional, the set of distributions that satisfy the discreteness constraint (D) is not convex. The remaining task is to show that the corresponding feasible set of I_G is compact and the objective function in (2) is continuous. This ensures the existence of an optimal solution.⁸

⁸We thank Yunus Aybas and Eray Turkel for sharing an earlier version of their working paper that provides a detailed proof on the existence of the solution. Although they impose the affine-closedness assumption on the value function, their proof applies to general regular value functions.

Lemma 2. *For any feasible J -pooling policy with $J \geq 3$, there exist a feasible bi-pooling policy that produces a (weakly) higher value of the objective function (2). Moreover, the maximum of (2) can be attained by some bi-pooling policy.*

3. Convex Value Functions

In this section, we consider the solution to program (1) when the value function u is convex. Below, we provide two examples which will be referred to throughout this paper.

Example 1 (Purchase Decision). The state θ is the value of a good, and the price of the good is p . The decision maker chooses to buy the good ($y = 1$) or not ($y = 0$). She designs an experiment to acquire information about θ . After the signals are observed but before the purchase decision is made, a cost shock η is realized, where η is distributed with density h on $[\underline{\eta}, \bar{\eta}]$, where $\underline{\eta} \leq -p$. When her posterior expectation of the state is x , she buys the good if and only if $x - p - \eta \geq 0$. Her interim utility function is her expected consumer surplus, $u(x) = \int_{\underline{\eta}}^{x-p} (x - p - \eta)h(\eta) d\eta$, which is a convex function with curvature $u''(x) = h(x - p)$.

Example 2 (Data Classifier). A decision maker knows that a random variable θ has distribution F on $[0, 1]$. She draws a large sample of this random variable, acquires some information about each unit through an experiment, and sorts these units into N bins. Units in the same bin will be treated as if they were identical. If the posterior mean of units in a bin is x , she chooses $y = x$ to maximize her interim expected utility $-(y - \theta)^2$. If the experiment generates a distribution G of posterior means, her ex ante expected utility is $\mathbb{E}_G[x^2] - \mathbb{E}_F[\theta^2]$. Her interim value function is given by $u(x) = x^2$, which has a constant curvature $u''(x) = 2$.

In the general information design problem, the designer faces a tradeoff between revealing information on the convex regions of her value function and suppressing information on the concave parts. The introduction of discreteness constraint raises another important question regarding how to disclose information efficiently on the convex region when the number of disposable “signal resources” is budgeted. Naturally, with convex value functions, the designer can increase her payoff by raising the informativeness of the experiment (i.e., by raising I_G). Consequently, it is without loss of generality

to focus on interval-partitional structures.⁹ Moreover the optimal experiment should always fully utilize her “signal resources” by setting $|\mathcal{K}_{I_G}| = N$.

Proposition 1. *Suppose u is convex. The optimal information structure G can be implemented by an interval-partitional structure with $|\mathcal{K}_{I_G}| = N$.*

Proof. The objective function (2) is the u'' -weighted area below I_G . Since u'' is non-negative, an information structure G' performs better than another information structure G if $I_{G'} \geq I_G$. Consider an arbitrary bi-pooling policy G such that two posterior means x_k^1 and x_k^2 are induced within the interval $[s_{k-1}, s_k)$ (see the left panel of Figure 3). We can construct another integral distribution function $I_{G'}$ by fixing the point $(x_k^1, I_G(x_k^1))$ and slightly rotating the piece of I_G on (x_k^1, x_k^2) in the counterclockwise direction until it hits the next piece of I_G on (x_k^2, s_k) (see the blue dashed piece in the left graph). By construction $I_{G'}$ is everywhere above I_G and everywhere below I_F . Moreover, the number of induced posterior means does not change. Therefore the information structure represented by $I_{G'}$ is feasible and produces a higher value of the objective function.

Next, suppose the optimal information structure induces $N' < N$ posteriors. Then the sender can divide some interval $[s_{k-1}, s_k)$ into two pieces $[s_{k-1}, s'_k) \cup [s'_k, s_k)$ (see the right panel of Figure 3). This new integral distribution with an additional kink point is everywhere higher than the original integral distribution, and will produce a higher value of the objective function when u is convex. \square

The optimality of interval-partitional policies with finite signal space is well understood in the economics literature. Dow (1991) establishes similar results in the context of a search model, as do Sørensen (1996) and Smith et al. (2021) in social learning models, and Wilson (1989) and McAfee (2002) in matching models.¹⁰ Note also that the proof of Proposition 1 relies on raising the integral distribution function, which implies a mean-preserving spread of the original distribution function. Therefore, Proposition 1 remains valid for any convex function u even when it is nondifferentiable.

⁹Ivanov (2021) also shows the optimality of interval-partitional information structures for convex value functions. Specifically, for fixed marginal distribution of the signals, he shows that any information structure which is not interval-partitional is majorized by an interval-partitional information structure.

¹⁰Some of the intuition for these results are consistent with lessons from optimal transport. See Villani (2003) and Galichon (2016) for standard textbooks on the topic.

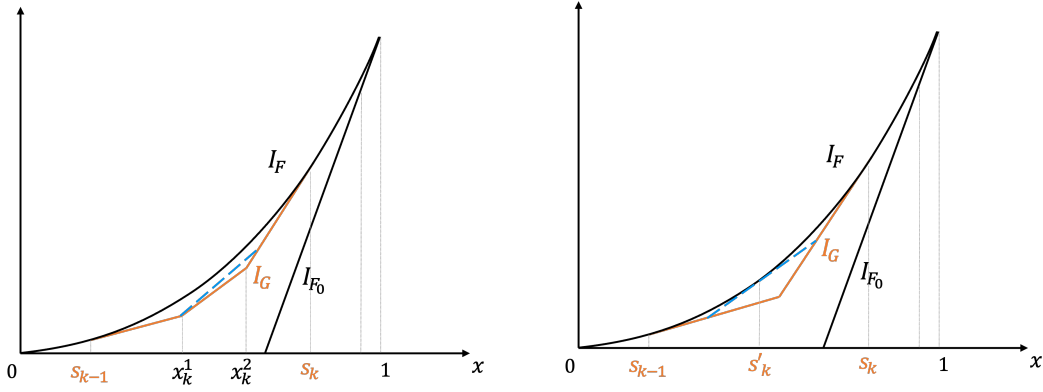


Figure 3: A rotation of I_G is profitable if the original information structure exhibits bi-pooling. Adding one more kink to I_G is profitable if the original information structure uses less than N signals.

Under an interval-partitional structure with N signals, the integral distribution I_G has $N + 1$ linear segments, with the i -th segment ($i = 1, \dots, N + 1$) tangent to I_F at s_{i-1} . Because the slope of I_F at s_{i-1} is $F(s_{i-1})$, we have

$$I_G(x) = \begin{cases} 0 & \text{if } x \in [0, x_1) \\ I_F(s_k) + F(s_k)(x - s_k) & \text{if } x \in [x_k, x_{k+1}) \text{ for } k = 1, \dots, N - 1 \\ I_F(1) + (x - 1) & \text{if } x \in [x_N, 1]. \end{cases} \quad (3)$$

For any $0 \leq a < b \leq 1$, let

$$\begin{aligned} \phi(a, b) &:= \mathbb{E}_F[t \mid t \in [a, b]], \\ \mu(a, b) &:= \mathbb{E}_{u'}[t \mid t \in [a, b]]. \end{aligned}$$

Here, $\mathbb{E}_{u'}$ is the conditional expectation operator using $u''(\cdot)/(u'(b) - u'(a))$ as the conditional density function. When u is strictly convex, this is a valid density of full support and the “conditional expectation” $\mu(\cdot, \cdot)$ is well-defined. Our characterization of the optimal information structure exhibits a remarkable dual relation.

Proposition 2. *Suppose u is strictly convex. The optimal information structure G , char-*

acterized by $\{s_k\}_{k=1}^{N-1}$ and $\{x_k\}_{k=1}^N$, must satisfy:

$$x_k = \phi(s_{k-1}, s_k) \quad \text{for } k = 1, \dots, N; \quad (\text{CE-F})$$

$$s_k = \mu(x_k, x_{k+1}) \quad \text{for } k = 1, \dots, N-1. \quad (\text{CE-u'})$$

Moreover, if both f and u'' are logconcave, the solution to this equation system is unique and characterizes the optimal information structure.

Proof. First, (CE-F) must hold by definition of an interval-partitional information structure. Substituting $\phi(s_{k-1}, s_k)$ for x_k in the I_G function in equation (3), we can express the objective function $\int_0^1 u''(x) I_G(x) dx$ as a function of $\{s_k\}_{k=1}^{N-1}$. The first-order necessary conditions for these interval cutoff points require

$$\int_{\phi(s_{k-1}, s_k)}^{\phi(s_k, s_{k+1})} u''(x)(x - s_k) dx = 0 \quad \text{for } k = 1, \dots, N-1.$$

Slightly rearranging terms, we obtain:

$$s_k = \frac{\int_{x_k}^{x_{k+1}} x u''(x) dx}{\int_{x_k}^{x_{k+1}} u''(x) dx} = \mu(x_k, x_{k+1}) \quad \text{for } k = 1, \dots, N-1.$$

Therefore, (CE-u') holds.

Let $\Gamma : [0, 1]^{2N-1} \rightarrow [0, 1]^{2N-1}$ be the mapping corresponding to the right-hand-side of (CE-F) and (CE-u'). Since Γ is monotone, Tarski's theorem implies that a largest (denoted $z' = (x'_1, s'_1, \dots, s'_{N-1}, x'_N)$) and a smallest (denoted $z'' = (x''_1, s''_1, \dots, s''_{N-1}, x''_N)$) fixed point exist. Since $z' \geq z''$, we must have $x'_1 - x''_1 \geq 0$. If $x'_1 - x''_1 = 0$, then the fact that $x_1 = \phi(0, s_1)$ and $\phi(\cdot)$ is strictly increasing implies that $s'_1 - s''_1 = 0$. Since $s_1 = \mu(x_1, x_2)$ and $\mu(\cdot)$ is strictly increasing, this further implies that $x'_2 - x''_2 = 0$. Iterating this argument then shows that $z' = z''$. If $x'_1 - x''_1 > 0$, the fact that $\phi(\cdot)$ is strictly increasing and satisfies property (5) stated in Lemma 3 below when f is logconcave implies that $s'_1 - s''_1 > x'_1 - x''_1$. Since a similar property as stated in Lemma 3 holds for $\mu(\cdot)$ when u'' is logconcave, this further implies that $x'_2 - x''_2 > s'_1 - s''_1$. Iterating this argument leads to $x'_N - x''_N > s'_{N-1} - s''_{N-1}$. But $x'_N - x''_N = \phi(s'_{N-1}, 1) - \phi(s''_{N-1}, 1) < s'_{N-1} - s''_{N-1}$ by Lemma 3, a contradiction. This shows that the largest and the smallest fixed point coincide. Because an optimal information structure exists and any candidate solution

must be a fixed point of Γ , the unique fixed point of Γ is the optimal solution. \square

Under an interval-partitional information structure, each signal x_k is the posterior expectation of the state conditional on the interval $[s_{k-1}, s_k]$. Proposition 2 states that under the optimal information structure, each s_k must be the expectation with respect to the distribution u' ,¹¹ conditional on the interval $[x_k, x_{k+1}]$. The two alternative expressions for the objective function, $\int_0^1 u(x) dG(x)$ in (1) and $\int_0^1 I_G(x) du'(x)$ in (2), are suggestive of a dual relationship between the distribution function and the marginal value function. To better understand this relation, observe that the first-order conditions for optimal cutoffs can be written as:

$$u(x_k) + u'(x_k)(s_k - x_k) = u(x_{k+1}) + u'(x_{k+1})(s_k - x_{k+1}). \quad (4)$$

Holding the neighboring interval cutoffs as given, a marginal increase in the cutoff s_k increases the chances of inducing posterior mean x_k as well as the value of x_k , as reflected by the terms involving $u(x_k)$ and $u'(x_k)$ on the left-hand-side of equation (4). At the same time, an increase in s_k reduces the chances of inducing posterior mean x_{k+1} and increases its value; the effects are given by the right-hand-side of (4). Considered as a function of s_k , the left-hand-side of (4) is linear and is tangent to $u(x)$ at x_k ; see the second line segment of \underline{u} in Figure 4(a). Similarly, the right-hand-side of (4) can be represented by the third segment of \underline{u} . At the optimal solution, the two marginal effects of an increase in s_k must balance each other, implying that these two line segments must intersect at s_k . That is, the piecewise-linear curve \underline{u} is tangent to u at x_k and at x_{k+1} , and is continuous with a kink at s_k .

Therefore we can identify a *minorant function*, \underline{u} , which is continuous, convex and piecewise linear such that \underline{u} is tangent to the value function u at the support of optimal information structure $\{x_k\}_{k=1}^N$.¹² The set of kink points of \underline{u} are the cutoffs of the optimal

¹¹We use u' here as a shorthand to stand for the distribution function $(u'(\cdot) - u'(0))/(u'(1) - u'(0))$.

¹²The optimal minorant function \underline{u} is reminiscent of the “price function” introduced by Dworczak and Martini (2019) and Kolotilin (2018). One difference is that the minorant function we obtain is everywhere below the value function, as opposed to the price function. This is because the existence and the continuity (at interval boundaries) of the price function are guaranteed by the strong duality in the unconstrained information design problem, while in our problem, the discreteness constraint breaks this strong duality. Nevertheless, one can view the minorant function as the dual payoff characterized using the techniques from optimal transport (Galichon, 2016; Smolin and Yamashita, 2022) if the action space is restricted to the set of actions induced under the optimal interval partition. It is also related to the virtual utility specified in Mensch (2021).

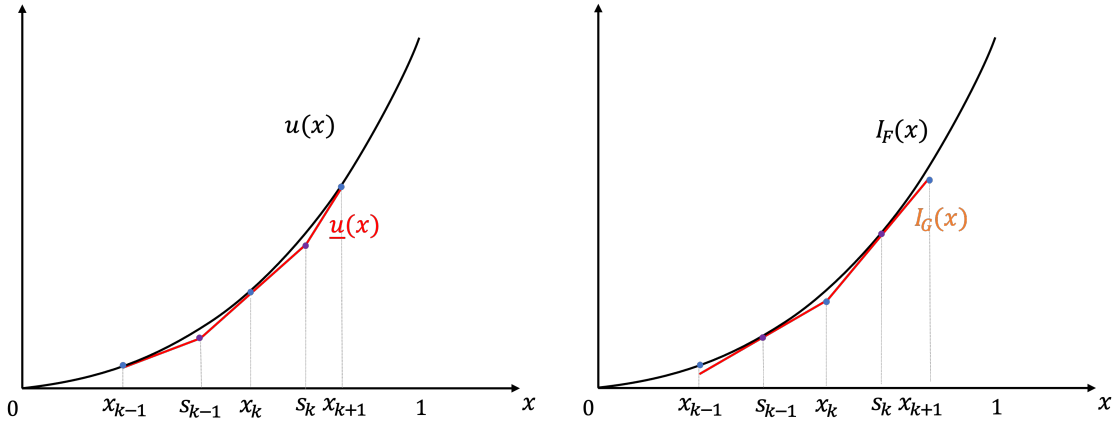


Figure 4: Dual expectations.

interval partition structure $\{s_k\}_{k=1}^{N-1}$. If we regard u as its “integral distribution function” of the “distribution” u' , then u bears a similar relationship to \underline{u} as I_F bears to I_G . By Corollary 1, we have already shown that each x_k in Figure 4(b) is the expectation of the state conditional on the interval $[s_{k-1}, s_k)$ under the distribution F . Using the same logic, the kink points of \underline{u} in Figure 4(a) must satisfy $s_k = \mathbb{E}_{u'}[\theta \mid \theta \in [x_k, x_{k+1})]$.¹³ In fact, (CE- F) and (CE- u') are also necessary conditions for general value functions where u' is a signed measure with a well-behaved Radon-Nikodym derivative with respect to Lebesgue measure and the conditional “expectations” are calculated accordingly, provided that the constrained-optimal information structure is indeed interval-partitional. See Section 4 for an application to S-shaped value functions and Section 9 for general discussions.

The uniqueness part in Proposition 2 relies on a property of logconcave densities. The role of such a property has been established and explored in the previous literature in quantization models (Mease and Nair, 2006) and cheap talk models (Szalay, 2012; Deimen and Szalay, 2019). Because we will further exploit this property for subsequent results, we state it here as a lemma for easier reference.

Lemma 3 (Mease and Nair (2006); Szalay (2012)). *If f is logconcave, then for any*

¹³This result generalizes many observations in previous applied models that involve coarse information. For example, Dow (1991) provides a similar characterization in a purchase game setup. Hopenhayn and Saeedi (2022) show that the optimal rating system sets the cutoffs at the midpoints between two adjacent posterior means when the supply function is linear. In their competitive market model, the supply function serves a similar role as our marginal value function u' .

$0 \leq a < b \leq 1$ and any $\varepsilon \geq 0$ such that $[a + \varepsilon, b + \varepsilon] \subseteq [0, 1]$,

$$\phi(a + \varepsilon, b + \varepsilon) \leq \phi(a, b) + \varepsilon. \quad (5)$$

Obviously, a similar property as stated in Lemma 3 holds for the $\mu(\cdot, \cdot)$ function if u'' is logconcave. When both f and u'' are logconcave, we refer to that as a *logconcave environment* in the following analysis.

3.1. Allocating scarce signal resources

The discreteness constraint in the information design problem imposes an upper bound on the size of the signal space. Proposition 1 shows that the optimal information structure partitions the state space into N subintervals. If a region of the state space is finely partitioned into more subintervals, then the experiment reveals finer details about that region of the state space as the receiver is more likely to distinguish across signal realizations. Given an experiment, we say a subinterval of the state space receives *closer scrutiny* than another if the width of this subinterval is smaller than that of another subinterval. Fixing the number of signals, if some region of the state space receives closer scrutiny, then some other region of the state space will receive less scrutiny. A natural question is: Which part of the state space should receive closer scrutiny in the optimal experiment?

In the simplest case, both f and u'' are uniform. Then $\phi(a, b) = \mu(a, b) = (a + b)/2$. Equations (CE- F) and (CE- u') imply that the state space $[0, 1]$ is divided into N equal-sized intervals with $s_k = k/N$ for $k = 0, \dots, N$; and the induced posterior means are evenly spaced, with $x_k = (2k - 1)/(2N)$ for $k = 1, \dots, N$. The optimal experiment gives every part of the state space equal scrutiny.

To investigate more complicated cases, denote the width of the k -th interval $[s_{k-1}, s_k)$ by $w_k = s_k - s_{k-1}$ for $k = 1, \dots, N$, and the distance between two adjacent posterior means by $d_k = x_{k+1} - x_k$ for $k = 1, \dots, N - 1$. Let $\{\Delta_i\}_{i=1}^{2N-1}$ be the interleaved sequence $\{w_1, d_1, w_2, \dots, d_{N-1}, w_N\}$. See Figure 5. We say that the sequence $\{\Delta_i\}_{i=1}^{2N-1}$ is *single-dipped* if it is decreasing then increasing or if it is monotone.

Proposition 3. *In a logconcave environment, $\{\Delta_i\}_{i=1}^{2N-1}$ is single-dipped.*

Proof. It suffices to show that $\Delta_i \geq \Delta_{i-1}$ implies $\Delta_{i+1} \geq \Delta_i$ for any $i = 2, \dots, 2N - 2$.

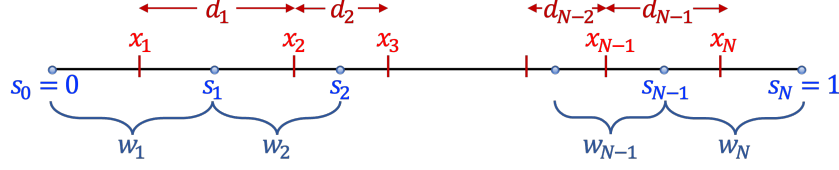


Figure 5: The interleaved sequence of Δ_i .

There are two cases to consider.

If i is even, then starting with the hypothesis that $d_k \geq w_k$ for some $k \leq N-1$, we want to show $w_{k+1} \geq d_k$. Suppose to the contrary that $d_k > w_{k+1}$. Then

$$x_{k+1} = \phi(s_k, s_{k+1}) = \phi(s_{k-1} + w_k, s_k + w_{k+1}) < \phi(s_{k-1} + d_k, s_k + d_k) \leq x_k + d_k = x_{k+1}.$$

The strict inequality obtains because ϕ is strictly monotone in each argument. The second inequality comes from Lemma 3. This is a contradiction.

If i is odd, then starting with the hypothesis that $w_k \geq d_{k-1}$ for some $k = 2, \dots, N-1$, we want to show $d_k \geq w_k$. Suppose to the contrary that $w_k > d_k$. Applying the similar argument above, we obtain

$$s_k = \mu(x_k, x_{k+1}) = \mu(x_{k-1} + d_{k-1}, x_k + d_k) < \mu(x_{k-1} + w_k, x_k + w_k) \leq s_{k-1} + w_k = s_k,$$

which is a contradiction. \square

An immediate consequence of Proposition 3 is that both the sequence of widths of subintervals, $\{w_i\}_{i=1}^N$, and the sequence of distances between adjacent posterior means, $\{d_i\}_{i=1}^{N-1}$, are single-dipped in a logconcave environment. The part of the state space where w_i attains the minimum is the region that receives the closest scrutiny in the optimal experiment. The result that the widths and distances are single-dipped implies there is a center of scrutiny—the subinterval with the minimum width—in the sense that the optimal experiment pays less and less scrutiny to states that are farther and farther away from this center. Intuitively, logconcavity of f and u'' implies that these two functions are single-peaked (Dharmadhikari and Joag-dev, 1998). In such environment, the two ends of the state space have low probability mass and low curvature. Therefore, information loss is less detrimental near the tails.

Define the index i^* such that $w_{i^*} = \min\{w_1, \dots, w_N\}$. The subinterval $[s_{i^*-1}, s_{i^*})$ is

the center of scrutiny of the optimal information structure. The next result concerns what determines this center.

Proposition 4. (i) *If u'' is a constant and f is single-peaked with mode m_f in the j -th interval, then $i^* \in \{j-1, j, j+1\}$.*
(ii) *If f is a constant and u'' is single-peaked with mode m_u in the k -th interval, then $i^* \in \{k-1, k, k+1\}$.*
(iii) *If f is single-peaked with mode m_f in the j -th interval and u'' is single-peaked with mode m_u in the k -th interval and $k \leq j$, then $i^* \in \{k-1, \dots, j+1\}$.*

Proposition 4 describes the sense in which the optimal experiment gives close scrutiny to states which have greater probability of occurrence and where the payoff is particularly sensitive to information. In the purchase decision example (Example 1), $u''(x) = h(x - p)$. If h is uniform and the prior density f has mode $m_f = 0.7$, then Proposition 4(i) suggests that the optimal experiment will give close scrutiny to states near 0.7, because such states have greater probability of occurring. The optimal experiment will resemble Experiment A in Figure 1 shown in the introduction. On the other hand, if f is uniform and h is single-peaked with mode 0 and $p = 0.3$, then u'' is single-peaked at $m_u = 0.3$. The purchase decision is particularly sensitive to belief when the posterior mean is near 0.3. The optimal experiment will resemble Experiment B in Figure 1. When both f and u'' are single-peaked, Proposition 4(iii) suggests that the optimal experiment will balance these two considerations. The exact location of the center of scrutiny will depend on details of f and h , but it is unlikely to be outside the interval $[0.3, 0.7]$.¹⁴

In the data classifier example (Example 2), the value function is quadratic and u'' is a constant. If the density f is strictly increasing, then Proposition 4(i) implies that the optimal information structure puts finer partitions in the upper part of the state space than in the lower part of the state space. Applied econometricians sometimes use an “equal-probability partition” to categorize continuous data so that each category contains the same expected number of observations. If the density function is increasing, the subintervals under this heuristic partition rule will also be narrower in the upper part of the state space than they are in the lower part. How does the optimal informa-

¹⁴Proposition 4(iii) does not exclude the possibility that $i^* = k-1$ or $i^* = j+1$, in which case the center of scrutiny will fall outside $[0.3, 0.7]$. This may occur, for example, if f is almost flat and h is highly asymmetric near its peak.

tion structure compare to the equal-probability partition?

Proposition 5. *Let the interval cutoffs under the optimal information structure be $\{s_k\}_{k=0}^N$ and those under the equal-probability partition be $\{s_k^{\text{eqp}}\}_{k=0}^N$. If f is weakly increasing and u'' is a constant, then for all k ,*

$$\frac{k}{N} \leq s_k \leq s_k^{\text{eqp}}.$$

When the density function f is increasing and u'' is constant, the optimal experiment is more likely to send higher signals than lower ones. Intuitively, the optimal information structure has finer partitions at the top, similar to what the equal-probability rule does, as higher states are more likely to be realized. However the equal-probability rule pays too much scrutiny to higher states relative to what the optimal rule requires when f is increasing and the loss is quadratic. Proposition 5 posits that under the specified environment, the optimal information structure lies in between the equal-probability partition and the equal-width partition.

3.2. Comparative statics

In this subsection, we present a series of comparative statics results as we vary the model primitives. Note that the dual relation identified in the previous section allows us to state these comparative statics results for both changes in the prior distribution f and the curvature of value function u'' .

Proposition 6. *In a logconcave environment, if either (a) the prior density changes from f to \hat{f} , with $\hat{f}(\cdot)/f(\cdot)$ being increasing; or (b) the value function changes from u to \hat{u} , with $\hat{u}''(\cdot)/u''(\cdot)$ being increasing, then all the interval cutoff points $\{s_k\}_{k=1}^{N-1}$ will increase and all the induced posterior means $\{x_k\}_{k=1}^N$ will also increase.*

Proof. We prove the case where f changes to \hat{f} . The proof for the change of value function is analogous. Since F increases according to the likelihood ratio order, the conditional distribution F on any interval $(a, b) \subseteq [0, 1]$ also increases, which implies that the conditional mean $\phi(a, b)$ becomes higher for any given a and b . In a logconcave environment the optimal $(x_1, \dots, x_N, s_1, \dots, s_{N-1})$ is the unique fixed point of Γ , the mapping corresponding to the right-hand-side of the equation system (CE- F) and (CE- u'). Since a likelihood ratio increase in F raises Γ , and Γ is monotone, a standard result in monotone comparative statics (Topkis, 1998) establishes that the any fixed point under \hat{f} is larger than the unique fixed point under f . \square

The qualification “logconcave environment” in Proposition 6 only requires f and u'' to be logconcave; it does not require \hat{f} or \hat{u}'' to be logconcave. In fact, the proposition holds so long as the solution to the system of equations (CE- F) and (CE- u') is unique either in the original environment or in the new environment. The comparative statics result with regard to the change of prior distribution is intuitive. A higher distribution (in the likelihood ratio order) means that the state is more likely to fall in the upper part of the state space. Naturally, the optimal experiment will induce more signals that reflect higher states. Szalay (2012), Chen and Gordon (2015), Deimen and Szalay (2023) and Smith et al. (2021) make similar observations in the context of cheap talk models and social learning models.¹⁵

Part (b) of Proposition 6 is more novel to literature and deserves some discussion. For two value functions u and \hat{u} , an increasing ratio of their corresponding curvatures \hat{u}''/u'' is equivalent to the condition that the marginal value function \hat{u}' is *more convex* than u' , i.e., there exists an increasing and convex function ψ such that $\hat{u}'(x) = \psi(u'(x))$.¹⁶ This condition implies that the designer’s value function under u on average changes faster in the upper region. Therefore, the designer has incentives to learn higher states more precisely as information is more valuable in higher states.

We may revisit the purchase decision example through the lens of Proposition 6. Suppose it is the *seller* of the good rather than the consumer who is designing the information structure.¹⁷ The price of the good is fixed at p and the seller wants to maximize the probability of making a sale. His interim value function is $\hat{u}(x) = pH(x - p)$, and therefore $\hat{u}''(x) = ph'(x - p)$. If the density h is nondecreasing, then \hat{u} is convex. Further, if h is logconcave, then $\hat{u}''(x)/u''(x) = ph'(x - p)/h(x - p)$ is decreasing in x . Our result shows that the induced posterior means $\{\hat{x}_k\}_{k=1}^N$ under the seller-optimal information structure are lower than the ones under the buyer-optimal information structure.¹⁸ Intuitively, the interests of the buyer and the seller are not aligned: the buyer cares

¹⁵Tian (2022) considers an interval division problem where the primitive is a cell function that maps an interval to a value. He establishes a comparative statics result similar to Proposition 6(a). Proposition 6(b) cannot be obtained from the cell function approach.

¹⁶In Hopenhayn and Saeedi (2022), the marginal value function for a seller is mathematically equivalent to their supply function. Hence our comparative statics result can partially explain their result that the optimal cutoffs are higher when the supply function is convex than when it is linear.

¹⁷In general, this model can encompass a large class of persuasion problems where a privately informed receiver has a binary action set (Kolotilin et al., 2017; Kolotilin, 2018; Candogan and Strack, 2023).

¹⁸Later we show in Section 4 that this conclusion remains valid when h is logconcave but not non-monotone, which causes the seller’s value function to become S-shaped.

about consumer surplus, which can be sensitive to high value realizations as they are likely to consume the good in such states, while the seller cares about the probability of trade, which is less sensitive to high states as the value of the good already well exceeds the price.

However, misalignment of interests by itself does not always result in different information designs if the curvatures of the value functions do not change. In Example 2, suppose the data classifier is a sender with utility $-(y - (\kappa_0 + \kappa_1 \theta))^2$. She commits to an experiment and reveals the experimental outcome (which bin the state belongs to) to a receiver who chooses action y to maximize $-(y - \theta)^2$. The receiver's interim value function is $u(x) = x^2$, while the sender's interim value function is $\hat{u}(x) = (2\kappa_1 - 1)x^2 + 2\kappa_0 x$. As long as $2\kappa_1 - 1 > 0$, the sender's value function is convex. Moreover, because $\hat{u}''(\cdot)$ is directly proportional to $u''(\cdot)$, the optimal information structure for the sender is identical to that for the receiver.¹⁹

Next, we investigate how the optimal information structure changes when the prior distribution becomes less variable. We say that a distribution with density \hat{f} is *uniformly less variable than* one with density f if $\hat{f}(\cdot)/f(\cdot)$ is unimodal (Whitt, 1985; Shaked and Shanthikumar, 2007).²⁰ This is a stronger notion than the standard variability ordering as it implies that variability ordering is preserved conditional on any subset of the state space. An example is that a normal distribution with a smaller variance is uniformly less variable than another normal distribution with a larger variance, while the means of the two distributions are not necessarily the same.

Proposition 7. *In a logconcave environment, if either (a) \hat{f} is uniformly less variable than f ; or (b) \hat{u}'' is uniformly less variable than u'' , then the corresponding sequence of optimal signals and interval cutoffs, $\{\hat{x}_1, \hat{s}_1, \dots, \hat{s}_{N-1}, \hat{x}_N\}$ crosses the original sequence $\{x_1, s_1, \dots, s_{N-1}, x_N\}$ at most once and from above.*

Proof. We prove the proposition for (a); the proof for (b) is similar. Suppose \hat{f}/f reaches a peak at $p \in (0, 1)$, and let k^* be the integer such that $s_{k^*} \leq p < s_{k^*+1}$. Define

¹⁹If $2\kappa_1 - 1 \leq 0$, the sender's value function is concave and he would choose to provide no information.

²⁰Define $S(\cdot)$ be the number of sign changes for a function. Whitt (1985) shows that \hat{f} is uniformly less variable than f if and only if the sign change counter $S(f - c\hat{f}) \leq 2$ for all $c > 0$, with equality for $c = 1$; and in the case of equality, the sign sequence is $+, -, +$.

the sequence,

$$\{\delta_j\}_{j=1}^{2N-1} := \{x_1 - \hat{x}_1, s_1 - \hat{s}_1, x_2 - \hat{x}_2, \dots, s_{N-1} - \hat{s}_{N-1}, x_N - \hat{x}_N\}.$$

To prove this proposition, it suffices to show that the two subsequences, $\{\delta_j\}_{j=1}^{2k^*+1}$ and $\{\delta_j\}_{j=2k^*+1}^{2N-1}$, are both single-crossing from below.

Let j be the first non-negative term in the subsequence $\{\delta_j\}_{j=1}^{2k^*+1}$. We want to show that $\delta_j \geq 0$ implies $\delta_{j+1} \geq 0$. Suppose j is odd; that is, $\delta_j = x_k - \hat{x}_k \geq 0$ and $s_{k-1} - \hat{s}_{k-1} < 0$ for some $k \in \{2, \dots, k^*\}$ (and if $j = 1$, then $k = 1$ and $s_{k-1} - \hat{s}_{k-1} = 0$). Then, letting $\hat{\phi}(\cdot)$ be the conditional mean function under \hat{f} , we have

$$x_k - \hat{x}_k = \phi(s_{k-1}, s_k) - \hat{\phi}(\hat{s}_{k-1}, \hat{s}_k) \leq \phi(s_{k-1}, s_k) - \phi(\hat{s}_{k-1}, \hat{s}_k) \leq \max\{s_{k-1} - \hat{s}_{k-1}, s_k - \hat{s}_k\},$$

where the first inequality follows because $\hat{f}(\cdot)/f(\cdot)$ is increasing on $[0, p]$, and the last inequality follows from Lemma 3. Since $s_{k-1} - \hat{s}_{k-1} \leq 0$, the above inequality implies $\delta_{j+1} = s_k - \hat{s}_k \geq \delta_j \geq 0$. Similarly,

$$s_k - \hat{s}_k = \mu(x_k, x_{k+1}) - \mu(\hat{x}_k, \hat{x}_{k+1}) \leq \max\{x_k - \hat{x}_k, x_{k+1} - \hat{x}_{k+1}\}.$$

Since $s_k - \hat{s}_k \geq x_k - \hat{x}_k$, the above inequality implies $\delta_{j+2} = x_{k+1} - \hat{x}_{k+1} \geq \delta_{j+1} \geq 0$. By induction, $\{\delta_{j+1}, \dots, \delta_{2k^*+1}\}$ are all non-negative. Similar reasoning applies to the case where j is even. Thus, we establish that $\{\delta_j\}_{j=1}^{2k^*+1}$ is single-crossing from below.

For the subsequence $\{\delta_j\}_{j=2k^*+1}^{2N-1}$, its single-crossing property is equivalent to the single-crossing property of the following:

$$\{\hat{x}_N - x_N, \hat{s}_{N-1} - s_{N-1}, \hat{x}_{N-1} - x_{N-1}, \dots, \hat{x}_{k^*+2} - x_{k^*+2}, \hat{s}_{k^*+1} - s_{k^*+1}, \hat{x}_{k^*+1} - x_{k^*+1}\}.$$

Note that $f(\cdot)/\hat{f}(\cdot)$ is increasing on $[p, 1]$. We apply a symmetric argument to prove the sequence listed above is single-crossing from below. Combining the single-crossing property of $\{\delta_j\}_{j=1}^{2k^*+1}$ and $\{\delta_j\}_{j=2k^*+1}^{2N-1}$ establishes the desired result. \square

An implication of Proposition 7 is that $\{\hat{x}_k\}_{k=1}^N$ crosses $\{x_k\}_{i=k}^N$ at most once and from above.²¹ If they indeed cross, this means that when the distribution becomes uniformly

²¹Proposition 7 does not exclude the possibility that these sequences do not cross. For example, if $\hat{f}(\cdot)/f(\cdot)$ is unimodal with a peak p close to 1, then changing the density from f to \hat{f} is almost like a

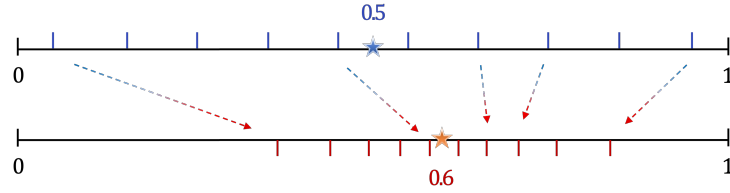


Figure 6: The induced posterior means of the optimal information structure are more compressed as the prior distribution becomes uniformly less variable. We use a constant u'' in this illustration. The prior distributions are truncated normal on $[0, 1]$ with parameters $(0.5, 1)$ for the upper panel and $(0.6, 0.1)$ for the lower panel.

less variable, there exists a k^* such that x_k increases for all $k < k^*$ and decreases for all $k \geq k^*$. See Figure 6 for an illustration. This figure shows that a uniformly less variable density will cause the induced posterior means to be more compressed.²²

3.3. Coarse mechanism design

The duality between integral distribution function and value function have implications beyond persuasion models. In particular we can adapt the minorant function approach to study mechanism design problems where the agent's payoff is linear in his private type and where the principal is constrained to use finite mechanisms. In this subsection, we illustrate this with a simple model of nonlinear pricing with finite menu.²³ This problem has been studied extensively by Bergemann et al. (2012, 2021) and Wong (2014). Our main objective is to demonstrate the equivalence between our coarse information design problem and a class of coarse mechanism design problems, and to leverage the results developed in earlier sections for application in the latter.

Consider the classic nonlinear pricing model of Mussa and Rosen (1978). A seller supplies products to a continuum of buyers with unit demand. Each buyer has private information about his preference $\theta \in [0, 1]$, distributed according to F with a continuous density function f . The seller designs a menu that specifies a set of quality and

likelihood ratio increase in the prior distribution, and it is possible that all interval cutoffs and induced posterior means increase.

²²Deimen and Szalay (2023) demonstrate a nested-interval property within a cheap talk model, assuming symmetric prior distributions with the same mean of 0. Under these conditions, the uniform variability order implies a decreasing likelihood ratio on the positive support and a reverse order on the negative support.

²³Bolton and Dewatripont (2005) provides several applications in contract theory that inherits similar structures, e.g. credit rationing, optimal taxation and monopoly regulation.

price pairs $\{(q_i, p_i)\}_{i \in \Sigma}$. A buyer with type θ who chooses a specific pair (q, p) from the menu receives the payoff $\theta v(q) - p$. Seller's production cost of supplying quality q is $c(q)$. For simplicity, we assume v and c are twice differentiable with $v' > 0$, $v'' < 0$, $c' > 0$, $c'' > 0$, and $v(0) = c(0) = 0$. Additionally, we impose the constraint that the seller can only offer menus with a finite number of options, i.e., $|\Sigma| \leq N$, introduced in Bergemann et al. (2012, 2021) and Wong (2014).²⁴ Throughout our discussion, we focus on the profit-maximizing menu for the seller; the analysis can be readily adapted to study welfare maximization by a planner.

By the revelation principle, we can focus on direct mechanisms $\{q(\theta), p(\theta)\}$, where $q(\cdot)$ and $p(\cdot)$ are step functions because the menu is constrained to be finite. In fact, we can further restrict our attention to (weakly) increasing allocation rule $q(\theta)$, as any such rule is implementable by adjusting the transfers $p(\theta)$. For each incentive compatible allocation $q(\theta)$, the seller's profit takes the well-known formula,

$$\int_0^1 \left[v(q(\theta)) \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) - c(q(\theta)) \right] dF(\theta). \quad (6)$$

Define $\varphi := \theta - (1 - F(\theta))/f(\theta)$ to be the virtual valuation, and let \tilde{F} be its associated distribution on $[0, 1]$. For simplicity, assume that virtual valuation is strictly monotone in type. If the pricing menu is unconstrained (i.e., not required to be finite), then (6) can be solved by maximizing the integrand pointwise. With some abuse of notation, we let $q^*(\psi) := \arg\max_{q \geq 0} \varphi v(q) - c(q)$ be the unconstrained optimal allocation. For $\varphi \in [0, 1]$, define $u(\varphi) := \max_q \varphi v(q) - c(q)$ to be the maximum profit that can be obtained from each virtual valuation type φ . Because the pointwise profit is linear in the consumer's virtual type, u is increasing and convex.

Let $q(\varphi)$ represent the allocation to virtual type φ in a finite menu, and define $\underline{u}(\varphi) := \varphi v(q(\varphi)) - c(q(\varphi))$ corresponding to such an allocation. The objective of the coarse mechanism design problem is to maximize $\int_0^1 \underline{u}(\varphi) d\tilde{F}(\varphi)$, subject to the restriction that $q(\cdot)$ is nondecreasing and takes at most N different values (other than 0). By definition, $\underline{u}(\varphi)$ is weakly below $u(\varphi)$ for any $\varphi \in [0, 1]$. Our goal is to show that the relationship between \underline{u} and u in the optimal finite menu problem is the same as

²⁴Wong (2014) uses the first-order approach to derive results for this problem. He mainly focuses on the marginal benefit from increasing the size of the menu, and discusses the approximation property when N grows large. Bergemann et al. (2021) study a linear-quadratic version of this model using the quantization approach.

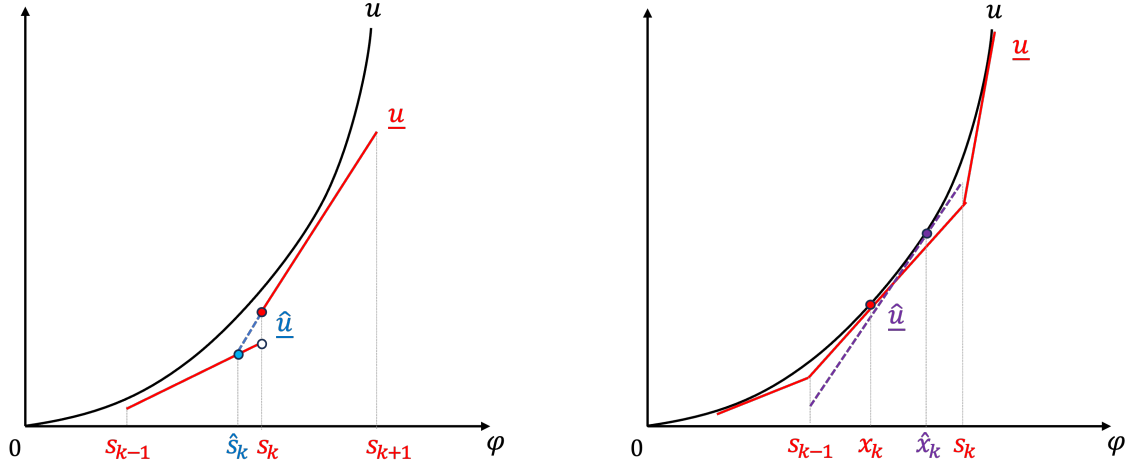


Figure 7: Properties of optimal \underline{u} .

that between the minorant function and the interim value function as depicted in Figure 4 for the coarse information design problem.

Any incentive-compatible menu with N options would imply \underline{u} must be piecewise affine with $N + 1$ pieces, including a horizontal piece arising from the outside option. To establish the equivalent characterization of the optimal finite menu and the optimal discrete information structure, it suffices to establish the following two observations: (a) the optimal \underline{u} is continuous and convex; and (b) each piece of the optimal \underline{u} on $[s_{k-1}, s_k)$ is tangent to u at $x_k = \mathbb{E}_{\tilde{F}}[\varphi \mid \varphi \in [s_{k-1}, s_k)]$.

For part (a), to prove continuity, suppose the \underline{u} function induced by some allocation $q(\cdot)$ has a discontinuity at some cutoff point s_k , as depicted in the left panel of Figure 7. Then we can adjust the transfers to construct another incentive-compatible allocation $\hat{q}(\cdot)$, which supplies a higher quality good to consumers with virtual values between \hat{s}_k and s_k while the allocation to other consumers are unchanged. The corresponding \hat{u} function (shown by blue dotted line for $\varphi \in [\hat{s}_k, s_k)$ in the left panel of Figure 7) for this modified allocation is everywhere higher than \underline{u} for the original allocation. This would yield a strictly higher profit, a contradiction. The convexity of the optimal \underline{u} then follows directly from continuity and from the fact that the slope of \underline{u} is equal to $v(q(\varphi))$, where $q(\cdot)$ must be nondecreasing.

For part (b), first note that each piece of \underline{u} is tangent to u at some point; otherwise we can adjust the allocation to raise \underline{u} and increase the seller's profit. Suppose, for

$\varphi \in [s_{k-1}, s_k]$, the tangency point of \underline{u} and u is x_k (shown in the right panel of Figure 7), and suppose $x_k \neq \mathbb{E}_{\tilde{F}}[\varphi \mid \varphi \in [s_{k-1}, s_k]]$. We can construct another allocation $\hat{q}(\cdot)$ such that the corresponding \hat{u} has a piece tangent to u at $\hat{x}_k = \mathbb{E}_{\tilde{F}}[\varphi \mid \varphi \in [s_{k-1}, s_k]]$ (shown as the purple dotted piece), and is otherwise equal to the original \underline{u} . Then,

$$\begin{aligned} \int_{s_{k-1}}^{s_k} \underline{u}(\varphi) d\tilde{F}(\varphi) &= [\tilde{F}(s_k) - \tilde{F}(s_{k-1})][u(x_k) + u'(x_k)(\hat{x}_k - x_k)] \\ &< [\tilde{F}(s_k) - \tilde{F}(s_{k-1})]u(\hat{x}_k) = \int_{s_{k-1}}^{s_k} \hat{u}(\varphi) d\tilde{F}(\varphi), \end{aligned}$$

where the inequality follows from the strict convexity of u . This contradicts the optimality of the allocation corresponding to original \underline{u} function.

Overall, our analysis implies that the optimal \underline{u} exhibits the same property as the minorant function that we introduce to explain the dual property. For $k = 1, \dots, N$, the optimal menu allocates goods with quality $q_k = q^*(x_k)$ to consumers with virtual types $\varphi \in [s_{k-1}, s_k]$, where $x_k = \mathbb{E}_{\tilde{F}}[\varphi \mid \varphi \in [s_{k-1}, s_k]]$ (with $s_N = 1$). Consumers with virtual type $\varphi \in [0, s_0]$ (and those with negative φ) are excluded and receive the outside option. Moreover, for $k = 0, \dots, N-1$, since the interval cutoff s_k is a kink-point of \underline{u} , we have $s_k = \mathbb{E}_{u'}[\varphi \mid \varphi \in [x_k, x_{k+1}]]$ (with $x_0 = 0$). Hence, the dual expectations characterization is valid for the finite-menu nonlinear pricing problem, and Propositions 3, 6, and 7 continue to hold (where the prior distribution is interpreted as the distribution of virtual type). The following example illustrates how we can study the structure of optimal finite menu by simply checking the properties from the optimal unconstrained menu and the associated value function.

Example 3. Suppose $v(q) = q^\beta$ and $c(q) = \gamma q$ for $\beta < 1$. Then the unconstrained optimal allocation for the seller is $q^*(\varphi) = (\beta\varphi/\gamma)^{1/(1-\beta)}$, and the “value function” for this problem is $u(\varphi) = (1-\beta)(\gamma/\beta)(\beta\varphi/\gamma)^{1/(1-\beta)}$. Let the value of the good change to $\hat{v}(q) = q^{\hat{\beta}}$, with $\hat{\beta} > \beta$. Note that $\hat{v}(q) > v(q)$ for all $q > 1$.²⁵ The value function $\hat{u}(\varphi)$ corresponding to the higher valuation of quality satisfies the property that $\hat{u}''(\cdot)/u''(\cdot)$ is increasing. If the density of virtual valuation is logconcave on $[0, 1]$, this setting is a log-concave environment. By the same logic leading to Proposition 6, all the interval cutoff points will move to the right when the benefit increases from $v(q)$ to $\hat{v}(q)$. In particular,

²⁵The seller’s optimal menu will indeed offer $q_k > 1$ for all k when the marginal cost γ is sufficiently low.

we have $\hat{s}_0 > s_0$. This means that the seller’s optimal menu will exclude a larger set of consumers, despite the fact that the good has become more valuable to all consumers. Intuitively, increase in β not only implies higher valuation for quality, but also means greater importance to screen the high virtual value buyers. When the screening device is constrained, the optimal finite menu naturally involves more discrimination towards high-end customers, leading to the exclusion of more low-end customers. Note that this is a novel feature for finite menu design, as the coverage of customer in unconstrained problems only depends on the distribution of virtual value.

4. S-shaped Value Functions

The analysis in the previous section is based on the premise that the interim value function u is convex, which is always the case where the designer wishes to transmit as much information as possible, such as in the case of single-agent information acquisition problems. In other contexts such as Bayesian persuasion, an agent designs an information structure to influence the action chosen by another agent, and the value function is not necessarily convex. The general coarse information design problem for arbitrary value function u is complicated as we cannot rule out the possibility that a bi-pooling structure may be optimal.

In this section, we extend our analysis to S-shaped value functions. A function u is *S-shaped* if there exists some $\hat{x} \in (0, 1)$ such that u is convex on $[0, \hat{x})$ and concave on $(\hat{x}, 1]$. In other words, u'' is single-crossing from above.²⁶ S-shaped value functions have been extensively studied in persuasion literature because they capture a range of economic applications. The solution to persuasion problems involving this class of functions are easy to characterize. In particular, there is tension between providing information on the convex region and suppressing it on the concave region. The unconstrained optimal policy features “upper-censorship,” namely the optimal experiment without the discreteness constraint will reveal full information on $[0, s^*]$ and coarsen information by pooling all states in $(s^*, 1]$ for some $s^* \leq \hat{x}$ (Kolotilin et al., 2022).²⁷ The optimal unconstrained information structure G^* has an integral distribution I_{G^*} that coincides with I_F everywhere to the left of s^* ; then becomes linear with slope $F(s^*)$ to the right of s^* until it intersects with I_{F_0} , and then coincides with I_{F_0} to the right of the

²⁶The analysis in this section also extends to the case where u'' is single-crossing from below.

²⁷It is possible that under some prior distributions, the optimal information structure contains no information.

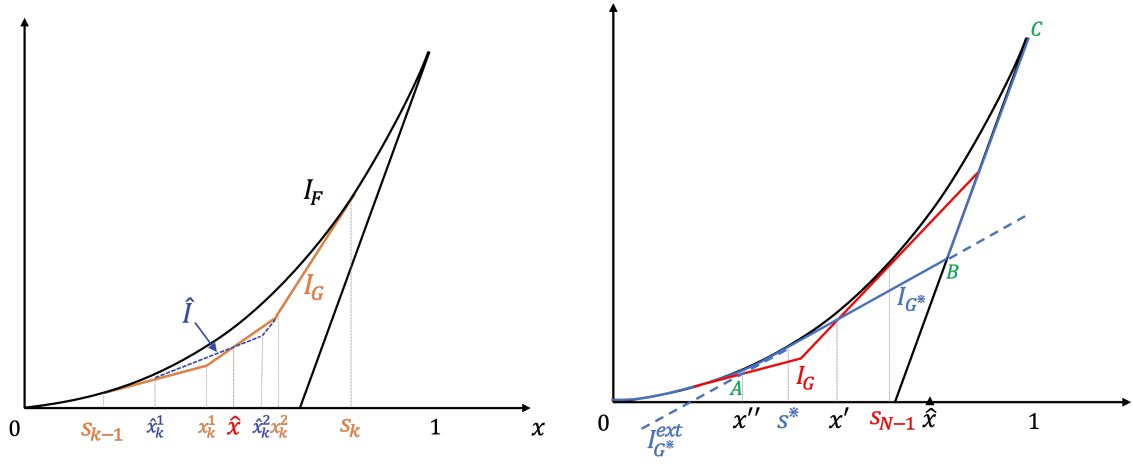


Figure 8: In the left panel, a clockwise rotation of the integral distribution function around the inflection point \hat{x} would raise the value of the objective function. In the right panel, the solid blue curve I_{G^*} represents the optimal unconstrained information structure. If I_G is a feasible discrete information structure with $s_{N-1} > s^*$, another feasible information structure \hat{I} which coincides with I_G to the left of x'' and follows the line segments ABC to the right of x'' will increase the value of the objective function.

intersection point. See the blue solid curve in Figure 8(b).

Proposition 8. Suppose u is S-shaped and the optimal information structures with and without the discreteness constraint (D) are G and G^* , respectively, where G^* features “upper-censorship” with $s^* \in (0, \hat{x}]$. Then,

- (i) G has an interval-partitional structure with $|\mathcal{K}_{I_G}| = N$;
- (ii) G is less informative than G^* .

Proof. (i) Suppose the optimal solution is a bi-pooling policy G such that two posterior means x_k^1 and x_k^2 are induced within the interval $[s_{k-1}, s_k)$. If $\hat{x} \geq x_k^2$ or $\hat{x} \leq x_k^1$, we can use the same argument as in the proof of Proposition 1 to show that the policy is suboptimal. If $\hat{x} \in (x_k^1, x_k^2)$, we can construct another integral distribution function \hat{I} by the following procedure. First, clockwise rotate the piece of I_G on (x_k^1, x_k^2) around the point $(\hat{x}, I_G(\hat{x}))$. Then we extend the new piece until it hits I_G at $\hat{x}_k^1 < x_k^1$ and hits the extended line of the next piece of I_G at $\hat{x}_k^2 > x_k^2$. See Figure 8(a) for a graphical illustration. By construction, $\hat{I} > I_G$ on (\hat{x}_k^1, \hat{x}) and $\hat{I} < I_G$ on (\hat{x}, \hat{x}_k^2) . Since $u''(x) > 0$ on (\hat{x}_k^1, \hat{x}) and $u''(x) < 0$ on (\hat{x}, \hat{x}_k^2) , we must have $\int_0^1 u''(x)I_G(x)dx < \int_0^1 u''(x)\hat{I}(x)dx$, a contradiction. The argument for $|\mathcal{K}_{I_G}| = N$ is similar to the convex case and therefore

omitted here.

(ii) Because I_{G^*} coincides with I_F to the left of s^* , and G has an interval-partitional structure by part (i), G is less Blackwell-informative than G^* if and only if $s_{N-1} \leq s^*$. Note that s_{N-1} cannot exceed \hat{x} , for otherwise we could rotate I_G clockwise at \hat{x} to increase the value of the objective function (2).

Suppose G is not less informative than G^* , i.e., $s_{N-1} \in (s^*, \hat{x}]$. Note that $I_G(s_{N-1}) = I_F(s_{N-1}) > I_{G^*}(s_{N-1})$ and $I_G(s^*) \leq I_F(s^*) = I_{G^*}(s^*)$. Therefore there exists $x' \in [s^*, s_{N-1})$ such that I_G crosses I_{G^*} from below at x' . Moreover, let $I_{G^*}^{\text{ext}}(x) := I_{G^*}(s^*) - F(s^*)(s^* - x)$ for $x \in [0, 1]$ to be the extrapolation of the linear segment of I_{G^*} (see Figure 8(b)). Then we have $I_G(0) > I_{G^*}^{\text{ext}}(0)$ and $I_G(s^*) \leq I_{G^*}^{\text{ext}}(s^*)$. Therefore there exists $x'' \in (0, s^*)$ such that $I_{G^*}^{\text{ext}}$ crosses I_G from below at x'' .

Consider an alternative informal structure with integral distribution \hat{I} given by:

$$\hat{I}(x) = \begin{cases} I_G(x) & \text{if } x < x'' \\ I_{G^*}^{\text{ext}}(x) & \text{if } x \in [x'', x'] \\ I_{G^*}(x) & \text{if } x > x'. \end{cases}$$

Note that $\hat{I} \in \text{ICPL}$ with $|\mathcal{K}_{\hat{I}}| \leq |\mathcal{K}_{I_G}|$. Thus \hat{I} is a feasible discrete information structure. By construction, $\hat{I}(x) > I_G(x)$ for $x \in [x'', x']$. Since $x' < s_{N-1} \leq \hat{x}$, we have $u'' > 0$ for $x \in [x'', x']$. This gives

$$\int_0^{x'} u''(x) \hat{I}(x) dx > \int_0^{x'} u''(x) I_G(x) dx.$$

Moreover, another information structure \tilde{I} which coincide with I_{G^*} on $[0, x')$ and coincide with I_G on $[x', 1]$ is feasible in the unconstrained information design problem without the discreteness constraint. Therefore,

$$\int_{x'}^1 u''(x) \hat{I}(x) dx = \int_{x'}^1 u''(x) I_{G^*}(x) dx \geq \int_{x'}^1 u''(x) \tilde{I}(x) dx = \int_{x'}^1 u''(x) I_G(x) dx,$$

where the inequality follows by revealed preference. Combining these two inequalities shows that \hat{I} is strictly better than I_G , a contradiction. \square

The result that the optimal experiment has an interval-partitional structure even

when u is S-shaped follows a similar logic as that for Proposition 1. If the information structure exhibits bi-pooling in a subinterval that contains \hat{x} , rotating the integral function clockwise at \hat{x} will increase payoff because u'' is positive to the left of \hat{x} and negative to the right of \hat{x} . Part (ii) of Proposition 8 shows that I_G must be everywhere below I_{G^*} ; therefore the optimal discrete information structure is less informative than the optimal unconstrained experiment in the sense of Blackwell (1953).

Once we establish the optimality of interval-partitional structure, we can apply our dual expectations characterization to solve the problem. One subtlety is that since now u'' is negative on some region, the object $\mu(\cdot)$ in (CE- u') is the barycenter instead of the conditional expectation. We provide a more thorough discussion on this point in the next section. Another issue is that in our discussion of the purchase decision example in Section 3.2, we assume that the value function \hat{u} for the seller is convex. If the density h of the cost shock is logconcave but not nondecreasing, then $\hat{u}''(x) = ph'(x-p)$ is single-crossing from above, and so \hat{u} is an S-shaped value function. For such value function, we need to provide an additional argument to establish Proposition 6. We offer such an argument in the appendix. Our earlier conclusion that the induced posterior means under the seller-optimal information structure are lower than the ones under the buyer-optimal structure remains valid even when the seller's value function \hat{u} is S-shaped.

5. General Value Functions

The coarse information design problem with convex or S-shaped value functions is relatively tractable because interval-partitional information structures are optimal. With general value functions, it is possible that the optimal information structure entails bi-pooling policies, and the analysis would become more involved. Nevertheless, in this section we show that the dual expectations property between interval cutoffs and induced posterior means still holds, albeit in a modified way.

When the value function u is not convex, u'' may be negative and therefore cannot be a valid density. Nevertheless it can be interpreted as the “density” corresponding to a *signed measure*. For some measurable set A under signed measure u' , its *barycenter* is given by

$$\frac{\int_A \theta u''(\theta) d\theta}{\int_A u''(\theta) d\theta}.$$

We can therefore interpret $\mu(a, b)$ in equation (CE- u') as the barycenter of set $A = [a, b)$

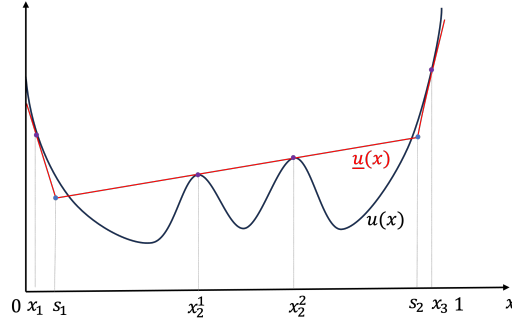


Figure 9: Optimal information structure for a general value function.

under signed measure u' . When the optimal information structure under general value function is interval-partitional, it is still characterized by the system of equations (CE- F) and (CE- u') stated in Proposition 2. The dual expectations property continues to hold, with “expectations” interpreted as “barycenter.”

The case where the information structure may not be interval-partitional is illustrated in Figure 9. The information structure represented by $\underline{u}(x)$ in this figure partitions the state space into three intervals: $[0, s_1]$, $[s_1, s_2]$ and $[s_2, 1]$.²⁸ The first interval induces signal x_1 and the third interval induces signal x_3 . The second interval induces two possible signals, x_2^1 and x_2^2 . If bi-pooling is optimal on $[s_1, s_2]$, then \underline{u} must be tangent to the value function u at the two induced posterior means. Such bi-tangency requirement pins down x_2^1 and x_2^2 . For given x_2^1 and x_2^2 , our characterization for the remaining posterior means and cutoff points works as follows: the cutoffs s_1 and s_2 are the respective barycenters of $[x_1, x_2^1)$ and $[x_2^2, x_3)$ under the signed measure u' , and x_1 and x_3 are the conditional means of $[0, s_1)$ and $[s_2, 1]$, respectively, under the prior distribution. Moreover, one can check the feasibility of such bi-pooling information structure by verifying whether the interval $[s_1, s_2]$ so obtained is compatible with the two given signals x_2^1 and x_2^2 . The necessary and sufficient condition for such feasibility are described by Lemma 1 in Arieli et al. (2023).

Proposition 9. *Suppose an information structure $I_G \in \text{ICPL}$ is optimal, with interval cutoff points $s_0 < \dots < s_K$ such that I_G is tangent to I_F at these points.*

- (i) *If $J_k = 1$, then $x_k = \mathbb{E}_F[\theta | \theta \in [s_{k-1}, s_k]]$.*
- (ii) *If $J_k = 2$, then $x_k^1 < x_k^2$ and they lie on an affine line bi-tangent to $u(x)$.*
- (iii) *Each cutoff s_k is the barycenter of $[y_k, y_{k+1})$ under the signed measure u' , where y_k*

²⁸For general value functions, $\underline{u}(x)$ is not necessarily everywhere below or above the value function.

is equal to x_k (if $J_k = 1$) or x_k^2 (if $J_k = 2$) and y_{k+1} is equal to x_{k+1} (if $J_{k+1} = 1$) or x_{k+1}^1 (if $J_{k+1} = 2$).

Proposition 9 only gives necessary conditions for an optimal information structure. Nevertheless, the set of information structures that satisfies those conditions is finite. In principle, an optimal solution can be derived by comparing the associated payoffs.

6. Discussion

6.1. Another application: design of energy efficiency ratings

Providing accessible information on the energy efficiency of electrical appliances or automobiles can assist households in estimating the costs associated with energy usage, enabling them to make informed decisions. However, energy consumption often involves externalities, meaning that the sole objective of reducing energy costs may not fully align with broader policy objectives that encompass environmental and other considerations. The development of suitable energy efficiency ratings must take into account these diverse concerns.

Let θ be a random variable which is uniformly distributed on $[\underline{\theta}, 1]$ (where $\underline{\theta} \in (0, 1)$). This random variable represents the energy efficiency of a car or other energy-consuming devices. In the case of gasoline vehicles, θ could represent parameters such as fuel economy or CO₂ emissions. In the case of lightbulbs, θ could represent electricity consumption measured in kilowatt-hours per thousand hours of use.

Suppose a household drives a car for y miles. Total energy consumption would be $y\theta$. The unit cost of fuel consumption is p . The household has quasi-linear preference in money and CARA utility from car usage,

$$U(y, \theta) = 1 - e^{-y} - py\theta.$$

Given an estimation of the expected energy efficiency x , the household's optimal usage is $y(x) = -\log(px)$. Thus, its indirect value function is $u(x) = 1 - e^{-y(x)} - py(x)x$, which is convex. Now consider a government that not only cares about the household's welfare, but also wants to reduce total energy consumption for environmental reasons and reduce private car usage for traffic management. Its indirect value function is $\hat{u}(x) = u(x) - \lambda_1 y(x)x - \lambda_2 y(x)$, where $\lambda_1, \lambda_2 > 0$ measure the intensity of the two

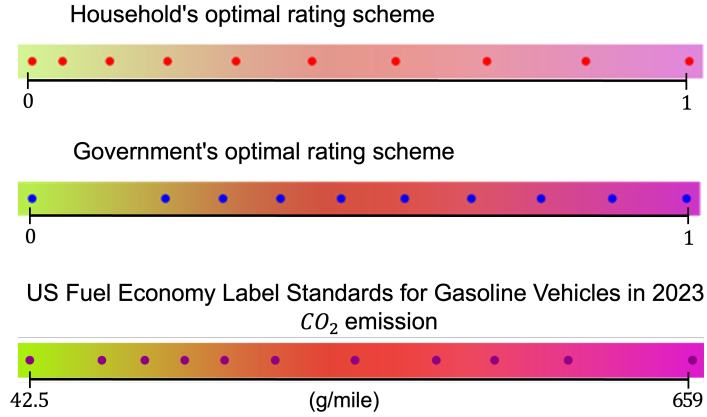


Figure 10: The first and second panel are the optimal ratings scheme for the household and for the government calculated based on the parameters $\underline{\theta} = 0.01$, $p = 0.1$, $\lambda_1 = 0.3$, $\lambda_2 = 0.05$, and uniform prior distribution. The third panel shows the CO₂ emissions standard for vehicles in the U.S. in 2023 (Source: <https://www.fueleconomy.gov/feg/label/learn-more-gasoline-label.shtml>).

respective objectives. Note that \hat{u} is concave to the left of $\lambda_2/(\lambda_1 + p)$ and is convex to the right of it.

The bottom panel of Figure 10 presents the ratings standards of fuel economy labels for gasoline vehicles in the U.S. One observation is that the width of the first class (representing the most efficient vehicles) is wider than that of the second one. A similar pattern can be found in the ratings of electrical devices such as washing machines and the energy ratings scheme for residential buildings in the EU. Remarkably, we find the same pattern in the numerical solutions for the government's optimal information structure. Because the government's value function is concave near $\underline{\theta}$ (assuming $\underline{\theta} < \lambda_2/(\lambda_1 + p)$), the first interval of the optimal ratings scheme is relatively wide. Intuitively, when energy performance is very efficient, usage is very sensitive to efficiency (as $dy(x)/dx = -p/x$ is large in absolute value when x is small). Thus, a coarser interval around the smallest values of θ can help achieve the government's objectives of reducing usage and energy consumption.²⁹ Moreover, because \hat{u}''/u'' is increasing, the cutoff points under the government-optimal ratings scheme are uniformly higher than those under the household-optimal ratings scheme.

²⁹Though we use a uniform prior distribution for the numerical examples, the observation that the first interval is coarser is robust across different years in 2017–2024 (the U.S. Environmental Protection Agency changes the rating scheme every year as the underlying distribution of efficiency varies across years).

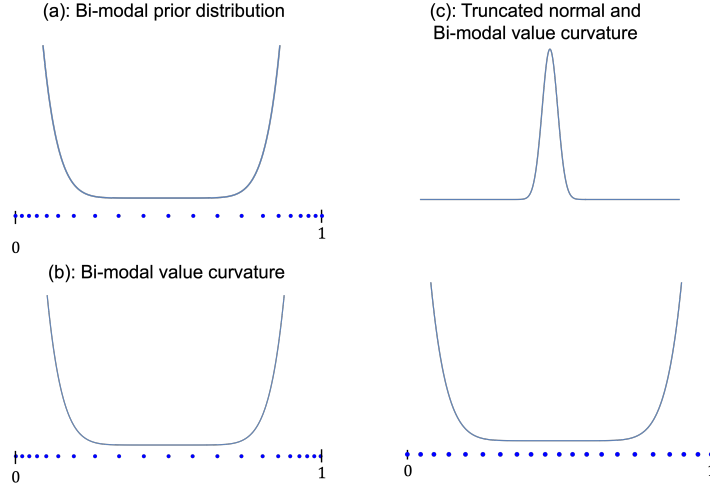


Figure 11: Numerical solutions for different prior distributions and value functions. In (a), $f(x) = 2048(x - 0.5)^2$ and $u''(x) = 2$. In (b), $f(x) = 1$ and $u''(x) = (x - 0.5)^8$. In (c), the prior distribution is truncated Normal on $[0, 1]$ with parameters $(0.5, 0.03)$, and $u''(x) = (x - 0.5)^8$.

6.2. Beyond logconcave environments

The conditions (CE- F) and (CE- u') for interval cutoffs and induced posterior means are necessary but not sufficient for optimality. When the solution to that equation system is unique, the necessary conditions are also sufficient. A logconcave environment ensures uniqueness of the solution to (CE- F) and (CE- u') and facilitates comparative statics analysis.³⁰ In this subsection, we provide numerical examples of situations where the curvature functions or the prior densities are not logconcave or not even single-peaked. In these examples the uniqueness of the solution to (CE- F) and (CE- u') can be numerically verified.³¹ Furthermore, the main message in our analysis—the decision maker gives closer scrutiny to states that are more likely to occur or states with higher instrumental value of information—remains robust.

In Figure 11(a), we use a bi-modal density function with modes at 0 and 1, while the value function is quadratic. The dotted line and the bottom of each panel plots the interval cutoffs of the optimal information structure. The decision maker gives the

³⁰When there are multiple solutions to (CE- F) and (CE- u'), the optimal solution need not be the largest or the smallest fixed point, thus rendering standard monotone comparative statics difficult.

³¹Starting with initial values near 1, an iterated solution to (CE- F) and (CE- u') converges to the largest fixed point; and starting with initial values near 0, it converges to the smallest fixed point. The solution is unique if the largest fixed point and the smallest fixed point coincide.

closest scrutiny to those states around 0 and 1 as they are most likely to be realized. In 11(b), we consider the uniform prior distribution, while u'' is bi-modal with modes at 0 and 1. Symmetrically, the optimal information structure gives close scrutiny to states around 0 and 1 as they have more impact over the decision maker's payoffs. In 11(c), u'' is still bi-modal, but the prior distribution is truncated normal distribution with a single mode at 0.5. In this case, the optimal information structure needs to compromise between the two forces, and gives close scrutiny to states around 0, 0.5 and 1. The widths of successive intervals are not single-dipped in these examples.

6.3. Connection with cheap talk

Our model of coarse information design bears some resemblance to the standard cheap talk model of Crawford and Sobel (1982), where information coarsening arises endogenously due to incentive compatibility constraints. In Crawford and Sobel (1982), to verify an interval-partitional information structure is an equilibrium strategy, we only need to verify that the cutoff types are indifferent between two adjacent actions taken by the receiver under two adjacent posterior means. Suppose, for example, that the sender's utility is $-(y - \kappa_1 \theta)^2$ and the receiver's utility is $-(y - \theta)^2$, where $\kappa_1 > 1$. Then, the cheap-talk equilibrium, given a fixed number of signals N , is determined by the following system of equations:

$$\begin{aligned} x_k &= \phi(s_{k-1}, s_k) & \text{for } k = 1, \dots, N; \\ s_k &= \frac{x_k + x_{k+1}}{2\kappa_1} & \text{for } k = 1, \dots, N-1. \end{aligned}$$

The second set of equations are the indifference conditions for the cutoff types. They are different from the optimality conditions (CE- u') for the coarse information design problem; in general, the dual expectations property does not hold in cheap talk models unless the sender and receiver have the same preferences.³² Nevertheless this system of equations is still a monotone contraction mapping if the prior density f is logconcave. This means that some of the methods developed in this paper are applicable to cheap talk models.³³ We can also reproduce Proposition 3 to show that there is a center of

³²The dual expectations property comes from the continuity of the minorant function at the interval cutoffs. That is, the cutoff types are indifferent in terms of the virtual utility rather than the real utility.

³³For example, Proposition 7 regarding the effect of a uniformly less variable distribution is also applicable to a quadratic cheap talk model. Deimen and Szalay (2023) independently discover and explore the implications of such distribution changes for cheap talk equilibrium.

scrutiny in this cheap talk model. The reason is that, with $\kappa_1 > 1$, sender’s bias is smaller when θ is small in magnitude than when it is large. Consequently the intervals are narrower for states closer to 0, and become wider farther away from 0 (Gordon, 2010; Deimen and Szalay, 2019, 2023). In contrast, in the coarse information design problem, the center of scrutiny is mainly determined by value function curvature and prior density of the state. We have seen in Example 2 that changes in κ_1 has no effect on the optimal information structure in our model.

Recent developments in cheap-talk games have also explored of optimal information design by the sender. This literature assumes that an uninformed sender commits to an experiment, and sends an unverifiable message about the experimental outcome. In a uniform-quadratic setting with constant bias κ_0 (Ivanov, 2010; Lou, 2023; Kreutzkamp, 2023), the sender’s problem can be transformed into a problem similar to ours, with an additional set of truth-telling constraints that require the distance between every two adjacent messages to be at least $2\kappa_0$, and with the number of messages N being endogenously determined through this constraint. As elaborated in Lou (2023) and Kreutzkamp (2023), this additional constraint makes bi-pooling a possible candidate solution despite the quadratic value function. In our data classifier example, N is exogenously fixed and there are no truth-telling constraints. As a result the optimal experiment has an interval-partitional structure.

7. Conclusion

The literature on Bayesian persuasion often assumes no restrictions on the set of experiments that can be chosen, making the information design problem trivial when the value function is convex (as the perfectly revealing experiment would be optimal). One approach that has been taken to relax this assumption is to introduce a posterior-separable cost of acquiring information that depends on the experiment chosen (Caplin and Dean, 2015; Matějka, 2015; Bloedel and Segal, 2021; Ravid et al., 2022); see Denti (2022) for a critical assessment of that approach. In this paper, we adopt an alternative track by imposing a discreteness constraint on the signal space to reflect the coarseness of information structures. We take information coarsening as given and consider the best way of doing it. This approach naturally leads to a research agenda that investigates which parts of the state space deserve most attention or scrutiny in information acquisition decisions. Our paper only takes a first stab at this research agenda by consid-

ering a simple environment with one-dimensional state where the belief about the state matters only through the mean. Many questions regarding sequential information acquisition, competitive and complementary constrained information provision, or coarse information design under more varied payoff or informational environments remain to be explored.

Appendix

Proof of Lemma 1. Fix some $I_G \in \text{ICPL}$ with a finite number of kinks. It can only have a finite number of tangency points to I_F . Because I_G is piecewise linear and I_F is strictly convex, $I_G(\cdot) < I_F(\cdot)$ whenever they are not tangent. Enumerate the set of tangency points in increasing order and let $\{s_k\}_{k=0}^K = \{x \in [0, 1] : I_F(x) = I_G(x)\}$, with $s_0 = 0$ and $s_K = 1$.

Allocate all the kink points of I_G into the interval $[s_{k-1}, s_k)$ that contains them. This process is well-defined as every kinks point must lie in the interior of $[s_{k-1}, s_k)$ for some $k = 1, \dots, K$. Within each interval $[s_{k-1}, s_k)$, enumerate the kinks point in increasing order to get the sequence $\{x_k^j\}_{j=1}^{J_k}$, where J_k is the number of kink points contained in $[s_{k-1}, s_k)$.

Now we claim that the J -pooling policy with $\{s_k\}_{k=0}^K$ and $\{\{x_k^j\}_{j=1}^{J_k}\}_{k=1}^K$ can indeed induce a distribution G that corresponds to the integral distribution function I_G . It suffices to show that for every $k = 1, \dots, K$, the conditional distribution of G restricted on $[s_{k-1}, s_k)$, denoted by $G|_{[s_{k-1}, s_k)}$, can be indeed induced by that of F on the same interval. We need to check two consistency conditions: for every $k = 1, \dots, K$, (a) $G(s_k) = F(s_k)$; and (b) $I_F|_{[s_{k-1}, s_k)} \geq I_G|_{[s_{k-1}, s_k)} \geq I_F|_{[s_{k-1}, s_k)}^0$, where $I_F|_{[s_{k-1}, s_k)}^0$ is the degenerate distribution on $[s_{k-1}, s_k)$ that puts probability mass one at the conditional mean $x_k = \mathbb{E}_F[\theta | \theta \in [s_{k-1}, s_k)]$.

Part (a) is easy to check as the tangency condition implies that $F(s_k) = I'_F(s_k) = I'_G(s_k) = G(s_k)$ for $k = 0, \dots, K$. To verify (b), for any $x \in [s_{k-1}, s_k)$ we write:

$$I_G|_{[s_{k-1}, s_k)}(x) = \int_{s_{k-1}}^x \frac{G(t) - G(s_{k-1})}{G(s_k) - G(s_{k-1})} dt = \int_{s_{k-1}}^x \frac{G(t) - F(s_{k-1})}{F(s_k) - F(s_{k-1})} dt. \quad (7)$$

It is immediate that the first inequality of (b) holds if and only if $I_F \geq I_G$ on (s_{k-1}, s_k) . For the second inequality, note that the integral function corresponding to the degenerate distribution is

$$I_F|_{(s_{k-1}, s_k)}^0(x) = \begin{cases} 0 & \text{if } x \in (s_{k-1}, x_k) \\ x - x_k & \text{if } x \in [x_k, s_k). \end{cases} \quad (8)$$

It is obvious that the second inequality of (b) holds for $x \in [s_{k-1}, x_k)$. For $x \in [x_k, s_k)$,

we multiply both (7) and (8) by $F(s_k) - F(s_{k-1})$ and the difference reduces to:

$$\begin{aligned}
& I_G(x) - I_G(s_{k-1}) - (x - s_{k-1})F(s_{k-1}) - (x - x_k)(F(s_k) - F(s_{k-1})) \\
&= [I_G(s_k) + (x_k - s_k)F(s_k) - I_G(s_{k-1}) - (x_k - s_{k-1})F(s_{k-1})] + [I_G(x) - I_G(s_k) - (x - s_k)F(s_k)] \\
&= [I_F(s_k) + (x_k - s_k)F(s_k) - I_F(s_{k-1}) - (x_k - s_{k-1})F(s_{k-1})] + [I_G(x) - I_G(s_k) - (x - s_k)I'_G(s_k)] \\
&= I_G(x) - I_G(s_k) - (x - s_k)I'_G(s_k) \geq 0,
\end{aligned}$$

where the first term in brackets in the third line is equal to 0 by definition of x_k as the conditional mean on $[s_{k-1}, s_k)$, and the last inequality follows because I_G is convex. This establishes (b), and therefore by Strassen's (1965) theorem the conditional distribution of G restricted on $[s_{k-1}, s_k)$ can be indeed induced by the prior on the same interval. \square

Proof of Lemma 2. Suppose an optimal information structure G induces $J_k \geq 3$ signals between (s_{k-1}, s_k) . By Lemma 1, the corresponding I_G exhibits $J_k \geq 3$ kinks between (s_{k-1}, s_k) . Then one can construct two functions $\bar{I}, \underline{I} \in \text{ICPL}$ with the following properties: (a) $I_F > \bar{I} > I_G > \underline{I} > I_{F_0}$ on (x_k^1, x_k^3) , and $\bar{I} = I_G = \underline{I}$ for all $x \notin (x_k^1, x_k^3)$; (2) $\mathcal{K}_{\bar{I}} = \mathcal{K}_{\underline{I}} = \mathcal{K}_{I_G}$; and (c) $I_G = \lambda \bar{I} + (1 - \lambda) \underline{I}$ for some $\lambda \in (0, 1)$ (see Figure 12). Such construction is feasible without violating (MPC) because in this region, I_G is strictly below I_F and we can always slightly adjust the slopes of each piece. Since the objective function in program (2) is a linear functional over set of feasible ICPL functions, I_G must be sub-optimal.

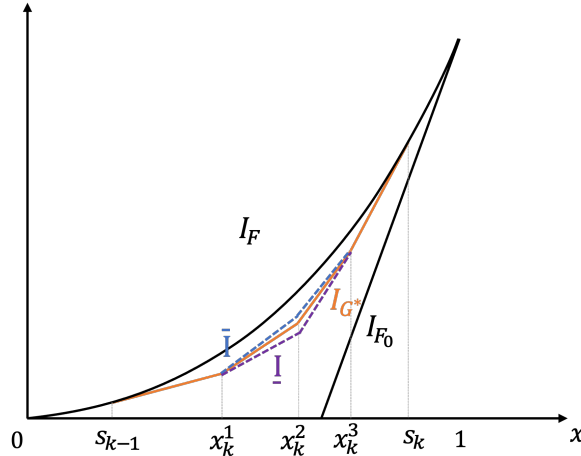


Figure 12

The proof for the existence of optimal signal structure directly comes from Theorem

4 in Appendix C, Aybas and Turkel (2021). In particular, it follows from the compactness of the feasible ICPL functions and continuity of the objective function in (2) with sup-norm. \square

Proof of Lemma 3. Mease and Nair (2006) and Szalay (2012) already discover this property. Here we present a simple proof with consistent notation for easy reference. Rewrite $\phi(a + \varepsilon, b + \varepsilon)$ and $\phi(a, b) + \varepsilon$ as

$$\begin{aligned}\phi(a + \varepsilon, b + \varepsilon) &= \int_{a+\varepsilon}^{b+\varepsilon} x \frac{f(x)}{F(b + \varepsilon) - F(a + \varepsilon)} dx = \int_a^b (x' + \varepsilon) \frac{f(x' + \varepsilon)}{F(b + \varepsilon) - F(a + \varepsilon)} dx', \\ \phi(a, b) + \varepsilon &= \int_a^b (x' + \varepsilon) \frac{f(x')}{F(b) - F(a)} dx' .\end{aligned}$$

The difference is

$$\int_a^b x \left(\frac{f(x + \varepsilon)}{F(b + \varepsilon) - F(a + \varepsilon)} - \frac{f(x)}{F(b) - F(a)} \right) dx.$$

Due to the logconcavity of f , the first density is dominated by the second in the likelihood ratio order. Hence, the difference is nonpositive. \square

Proof of Proposition 4. We prove (iii) first. The functions $\phi(\cdot)$ and $\mu(\cdot)$ are conditional expectations. Therefore, $\phi(a, b) \geq (a + b)/2$ and $\mu(a, b) \geq (a + b)/2$ for any $a < b \leq \min\{m_f, m_u\}$, because both f and u'' are increasing in the relevant region. An induction argument establishes that $w_1 \geq w_2 \geq \dots \geq w_{k-1}$. Similarly, $\phi(a, b) \leq (a + b)/2$ and $\mu(a, b) \leq (a + b)/2$ for any $\max\{m_f, m_u\} \leq a < b$, because both f and u'' are decreasing in the relevant region. An induction argument establishes that $w_N \geq w_{N-1} \geq w_{j+1}$. Together, they imply that there exists an $i^* \in \{k-1, \dots, j+1\}$ such that the sequence of widths attains a minimum at w_{i^*} .

A constant u'' is a special case of single-peaked u'' . We can arbitrarily set the mode of u'' at m_f , and part (i) follows. Similarly, if f is a constant, we can arbitrarily set its mode at m_u , and part (ii) follows. \square

Proof of Proposition 5. The proof of Proposition 4 shows that if u'' is a constant and f is increasing with mode $m_f = 1$, then $w_1 \geq \dots \geq w_N$. This implies that $k/N \leq s_k$.

To show the second inequality, let $q_k = F(s_k)$ for $k = 0, \dots, N$. By equation (CE- F),

$$x_k = \frac{\int_{s_{k-1}}^{s_k} t \, dF(t)}{F(s_k) - F(s_{k-1})} = \frac{\int_{q_{k-1}}^{q_k} F^{-1}(q) \, dq}{q_k - q_{k-1}},$$

where the second equality uses the change of variable $q = F(t)$. Note that $s_k = F^{-1}(q_k)$ and so $s_{k+1} - s_k = \int_{q_k}^{q_{k+1}} dF^{-1}(q)$. Applying integration by part,

$$\begin{aligned} x_k + x_{k+1} - 2s_k &= \frac{\int_{q_k}^{q_{k+1}} (q_{k+1} - q) \, dF^{-1}(q)}{q_{k+1} - q_k} - \frac{\int_{q_{k-1}}^{q_k} (q - q_{k-1}) \, dF^{-1}(q)}{q_k - q_{k-1}} \\ &\leq \int_{q_k}^{q_{k+1}} \frac{(q_{k+1} - q)}{q_{k+1} - q_k} \frac{1}{f(F^{-1}(q_k))} \, dq - \int_{q_{k-1}}^{q_k} \frac{(q - q_{k-1})}{q_k - q_{k-1}} \frac{1}{f(F^{-1}(q_k))} \, dq \\ &= \frac{q_{k+1} - q_k}{2f(s_k)} - \frac{q_k - q_{k-1}}{2f(s_k)}, \end{aligned}$$

where the inequality comes from f being increasing. When u'' is a constant, $s_k = \mu(x_k, x_{k+1}) = (x_k + x_{k+1})/2$. Therefore the left-hand-side of the inequality above is equal to 0, which implies $q_k - q_{k-1} \leq q_{k+1} - q_k$. This argument shows that

$$F(s_1) - F(0) \leq F(s_2) - F(s_1) \leq \dots \leq F(1) - F(s_{N-1}).$$

In contrast, under the equal-probability partition, we have

$$F(s_1^{\text{eqp}}) - F(0) = F(s_2^{\text{eqp}}) - F(s_1^{\text{eqp}}) = \dots = F(1) - F(s_{N-1}^{\text{eqp}}).$$

These two strings of inequalities imply that $F(s_k) \leq F(s_k^{\text{eqp}})$ for $k = 1, \dots, N-1$, and therefore $s_k \leq s_k^{\text{eqp}}$ for each k . \square

Proof of Proposition 6 (extended to the case when \hat{u} is S-shaped). Suppose u'' and f are both logconcave, and let $\hat{u}''(\cdot)/u''(\cdot)$ be decreasing. Suppose \hat{u} is an S-shaped value function, with $\hat{u}''(\cdot)$ changes from positive to negative at $\hat{x} \in (0, 1)$. Equations (CE- F) and (CE- u') still characterize the necessary conditions for optimality. We show that $\hat{s}_{N-1} \leq s_{N-1}$.

Take s_{N-1} as fixed at some value $s^\dagger \in (0, \hat{x})$. The right-hand-side of equations (CE- F) for $k = 1, \dots, N-1$ and (CE- u') for $k = 1, \dots, N-2$ defines a mapping $\Gamma^\dagger : [0, s^\dagger]^{2N-3} \rightarrow [0, s^\dagger]^{2N-3}$. The fixed point of this mapping gives the solution $(x_1, \dots, x_{N-1}, s_1, \dots, s_{N-2})$

to this equation system as a function of s^\dagger . Because Γ^\dagger is monotone and is increasing in s^\dagger , the largest fixed point is increasing in s^\dagger . Moreover, because $\hat{u}''(\cdot)/u''(\cdot)$ is decreasing and is positive on $[0, s^\dagger]$, monotone comparative statics implies $\hat{x}_k(s^\dagger) \leq x_k(s^\dagger)$ for all s^\dagger . Finally, define $x^\dagger(s^\dagger) := \phi(s^\dagger, 1)$, so that equation (CE-F) for $k = N$ is satisfied when $x_N = x^\dagger(s^\dagger)$. The remaining optimality condition is $s^\dagger = \mu(x_{N-1}(s^\dagger), x^\dagger(s^\dagger))$. We note that $\gamma(\cdot) := \mu(x_{N-1}(\cdot), x^\dagger(\cdot))$ is an increasing mapping. The optimal s_{N-1} must be a fixed point of γ . Since both $x^\dagger(\cdot)$ and $x_k(\cdot)$ are increasing functions and $\hat{x}_k(\cdot) \leq x_k(\cdot)$ (for $k = 1, \dots, N-1$), $\hat{s}_{N-1} \leq s_{N-1}$ would imply $\hat{x}_k \leq x_k$ for all k . Similarly, this would also imply $\hat{s}_k \leq s_k$ for all k .

Use $\hat{\mu}(\cdot)$ to denote the conditional mean function using \hat{u}'' as density, and $\tilde{\mu}(\cdot)$ to denote the conditional mean function using $\max\{\hat{u}'', 0\}$ as density. We have $\hat{\mu}(\cdot) \leq \tilde{\mu}(\cdot) \leq \mu(\cdot)$. Therefore,

$$\hat{\gamma}(s^\dagger) = \hat{\mu}(\hat{x}_{N-1}(s^\dagger), x^\dagger(s^\dagger)) \leq \mu(x_{N-1}(s^\dagger), x^\dagger(s^\dagger)) = \gamma(s^\dagger),$$

because $\hat{x}_{N-1}(s^\dagger) \leq x_{N-1}(s^\dagger)$. This implies that the largest fixed point of $\hat{\gamma}$ is smaller than the (unique) fixed point of γ , and therefore $\hat{s}_{N-1} \leq s_{N-1}$. \square

Proof of Proposition 9. Part (i) is by construction. Part (iii) can be obtained from the same first-orders condition as in Proposition 2. To show (ii), notice that the corresponding I_G supported on $[x_k^1, x_k^2]$ is completely below I_F . The optimality of such I_G implies that a local (clockwise or anti-clockwise) rotation at any point $(\tilde{x}, I_G(\tilde{x}))$ where $\tilde{x} \in [x_k^1, x_k^2]$ is not profitable. Denote the piece of I_G on $[x_k^1, x_k^2]$ as $I_G(\tilde{x}) + \beta(x - \tilde{x})$. Then the first-order necessary condition for objective function (2) with respect to β reduces to,

$$\int_{x_k^1}^{x_k^2} (x - \tilde{x})u''(x) dx = 0, \quad \text{for every } \tilde{x} \in [x_k^1, x_k^2].$$

Integration by part then implies,

$$u(x_k^1) - u'(x_k^1)(x_k^1 - \tilde{x}) = u(x_k^2) - u'(x_k^2)(x_k^2 - \tilde{x}), \quad \text{for every } \tilde{x} \in [x_k^1, x_k^2].$$

Graphically, this equation means that there exists an affine line that is bi-tangent to $u(\cdot)$ at both x_k^1 and x_k^2 . \square

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